Chapter 11
Borrow, Trade, Regroup, or Unpack? Revealing How Instructional Metaphors Portray Base-Ten Number

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Abstract This chapter uses embodied cognition to reveal unintended consequences for learning due to the processes that students enact with manipulatives. Base-ten block manipulatives and terms educators used for whole number arithmetic and place value are examples of ubiquitous "hands-on" instructional and assessment practices. Yet, the theoretical perspectives used to research this learning have not considered how students' actual physical movements represent intended ideas of arithmetic. The students whom educational researchers serve need us to better understand these practices in order to select and improve the design of such tools. Thus, this chapter examines how the language that educators use in combination with manipulatives influences students' understanding of addition and subtraction. This is the focus of the chapter for at least two reasons. First, it is crucial for elementary students to build procedural fluency and conceptual understanding of the base-ten number system. Second, these specific examples reveal the broader implications for any manipulative-based learning experiences for any topic across preK-16+ mathematics. Due to the physical motions students make during "hands-on" learning, it is critical to investigate these common practices through a lens of embodied mathematics learning. That is, research must attend to implications of how students move during instruction with "hands-on" materials as well as any metaphors educators orally express that imply motions even when students do not put their hands on materials.

Keywords Embodied cognition • Base-ten blocks • Metaphor • Physical movement • Dienes blocks • Digi-Blocks

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The intentions of a tool are what it does. A hammer intends to strike, …a lever intends to lift. They are what it is made for. … Sometimes in doing what you intend, you also do what the knife intends, without knowing.
—— Philip Pullman

The words and materials that educators choose to use to teach arithmetic are instructional tools intended to foster learning. Research is needed to understand how student learning outcomes with such tools reflect educators' intentions as well as how students' experiences and learning reveal unintended consequences. As the quote above implies, the focus of this chapter is to show that instructional tools used for base-ten number concepts, while in some ways accomplish the intended goals, may actually cut like a knife, that is, interfere with intended learning in ways and for reasons that until now have been unexplored. The goal of this chapter is to spark recognition of issues with using such tools through the lens of embodied cognition.

To understand the intended and unintended results of instruction with base-ten materials, the chapter first considers the intended learning, that is, base-ten number structure and meanings of addition and subtraction. Then some common instructional tools that educators have used to accomplish these ends will be shared before discussing how empirical and theoretical perspectives of embodied cognition can be used to posit potential unintended consequences of the ways students experience base-ten number with such tools.

Base-Ten Number Operations and Structure

The structure of the Hindu-Arabic or base-ten number system requires conceptual structures that are difficult for elementary students to develop. Using the position of numbers to represent different units of quantity where 0 represents none of a given unit was a significant societal advancement (West, Griesbach, Taylor, & Taylor, 1982). When children first learn to write numerals, they are unaware of this positional system. They simply understand that if a person means the quantity orally said as "twenty-six," they know it should be written as 26 (Fuson, 1992). This is not much different from knowing that if their name is Sara, they write it as *S-a-r-a* before understanding phonics. Consequently, when students begin writing larger numbers, they often write one hundred twenty-six, for example, as 10026 (Labinowicz, 1985). This written symbol reflects a logical understanding that they composed the quantities 100 and 26, but does not reflect the positional nature of the established written nomenclature.

It is only those with mathematically developed perspectives who see the base-ten number structure in the numeral 126 or 342, for example. Although many adults in the United States consider addition and subtraction to be basic math, consider the complexity of the mathematics underlying the base-ten number system:

- The system uses ten digits (0 through 9).
- The position of the digit determines its value (3 in 342 is different than 3 in 234).

- The face value of a digit is multiplied by its place value to determine its complete value (e.g., in 342, the face value of 3 is 3, so its complete value is 3 times its place value of 100, or 300 is its complete value).
- The system is multiplicative and additive ($342 = 3 \times 100 + 4 \times 10 + 2$).
- Each place value is ten times greater or less than the next.
- Each place value is determined by a power of base ten (e.g., 10^4).

This system is built upon and can only be fully understood by grasping all of these complexities. Yet, elementary students are expected to develop toward this full understanding long before they have even been introduced to multiplication or exponents, let alone mastered such topics. Due to the exponential structure of the base-ten number system, instructional practices that help students experience this structure are essential.

To accomplish this, many educators and researchers have investigated processes to help students restructure their conceptions of numbers as singular objects to see the units of quantities as higher-level units (Verschaffel, Greer, & Corte 2007). The units that can be expressed as powers of ten such as tens units, hundreds units, thousands units are composite units (Steffe & Cobb, 1988). In other words, drawing on psychological ideas of categorization, these are higher-level units, in that they are superordinate units in relation to a basic level unit (Rosch, Mervis, Gray, Johnson, & Boyes-Braem, 1976). In this case the basic level units are the ones. Different researchers have used a myriad of terms and labels for levels of base-ten thinking (Sarama & Clements, 2009). Consequently, I explain generally here the ways students think about numbers as they develop base-ten number understanding and emphasize the units students think about or see at each level that differs from the way adults may see these units. Students first think about quantities as values of single objects or ones units, although the students themselves at this point do not use a word like "ones," because this is a term that only becomes necessary as part of the larger base-ten place value system (Labinowicz, 1985). As adults who understand the place value system, however, it can be helpful to characterize students' thinking at this level as thinking only of the ones units. Through instruction, students begin to group quantities for efficiency and organization (e.g., counting 26 objects collectively by two as 2-4-6, etc.) but still think of the individual singles or ones units. Students can also learn to group objects into sets of ten, and additional ones such as two groups of 10 objects and 6 additional objects are 26 objects. Adults, however, often overestimate this ability to group, seeing it through their adult perspective as 2 tens units and 6 ones units. The students, however, need extensive time to develop that way of viewing hierarchical units to see the group itself as a unit the way adults may see as "a ten." What students first see in the same scenario are simply 10 objects, 10 objects, and 6 more objects. Even if students are able to parrot the language of "tens" at appropriate times, this does not mean they really think in terms of both the ten units and 10 ones units.

Students, who conceptually understand this positional place value system, can think flexibly about units to solve problems. Some examples of combinations of units for the number 342 can be thought about as:

(a) 342 ones
(b) 300 ones and 40 ones and 2 ones
(c) 3 hundreds units, 4 tens units, and 2 ones units (positional place value)
(d) 3 hundreds units, 3 tens unit, and 12 ones
(e) 34 tens and 2 ones

These ways portray just a few of the many ways these quantities can be composed and decomposed. Whereas both (a) and (b) use single units of thought as the item to be counted, (c), (d), and (e) all coordinate multiple levels of units. Formal positional place value is reflected in the (c) way of thinking about 342, yet example (d) shows how students should think of 342, if they need to subtract a number with more than 2 in the ones place using a traditional algorithm. The ability to think of such quantities structured as the ways shown in example (e) would mean they would not need to algorithmically divide 342 by 10 or use a memorized rule to move the decimal.

Base-Ten Manipulatives

To help students learn the culturally determined structure of the base-ten number system, many manipulative tools have been developed and are commonly used in schools. Some authors working within the tradition of radical constructivism suggest that the students should not be required to use manipulatives (Kamii, Lewis, & Kirkland 2001), whereas others suggest students can use such available tools as one of many student-determined ways to solve problems, which they consider consistent with constructivist approaches (Carpenter, Fennema, Franke, Levi, & Empson, 1999). In contrast, others claim such tools are crucial to learning (Fuson & Briars, 1990). These debates largely stem from and reflect differing theoretical perspectives of learning applied to issues of using manipulatives in general. In contrast, this chapter focuses on revealing the intended and unintended ways that specific manipulatives influence how students and teachers represent mathematical ideas.

The multiple materials used for teaching base-ten concepts can be categorized as *ungrouped* or *pre-grouped* and *proportional* or *nonproportional* models (Reys, Lindquist, Lambdin, & Smith, 2014; Van de Walle, Karp & Bay-Williams, 2010). Ungrouped or groupable models are individual objects (e.g., blocks, beans, sticks, and straws) that could be grouped in sets of ten but are not yet grouped and nothing inherent in the material structures that they be grouped this way (Fraivillig, 2017; Reys et al., 2014). Educators commonly use these ungrouped models during calendar math (Fraivillig, 2017). Even the phrasing of the standard 1.NBT.2a of the US Common Core State Standards for Mathematics implies proficiency with such ungrouped materials which are a learning goal by stating that "10 can be thought of as a bundle of ten ones — called a 'ten'" (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010, para. 1). Here I claim that this is an example of an instructional metaphor for how students will learn (bundling ungrouped materials) conflated with the intended mathematics (understanding tens units as composed of or containing 10 ones), because the intended learning is not that students know the particular context of a bundle but place value units.

The term pre-grouped models refer to how these individual models were prepared for instruction. For example, someone has already grouped some of these materials into sets of 10 for students to use during instruction along with the ungrouped items (Reys et al., 2014). Pre-grouped items could also refer to blocks that manufacturers molded to represent base-ten structure, such as blocks that are commonly referred to as "base-ten blocks" (Reys et al., 2014). Thus, these base-ten blocks might be more specifically referred to as prestructured, rather than simply grouped. Since this chapter will later show that "group" is an instructional metaphor, in the rest of the chapter, such materials will be referred to as prestructured.

The representations most researchers, teachers, and even national educational assessments (e.g., Warfield & Meier 2007) mean when referring to "base-ten blocks" are the specific most common type, which are Dienes blocks. The blocks were named after Zoltan Dienes, the mathematician who created them to help students represent arithmetic in multiple bases, including base ten (Web Minder, 2014). These blocks consist of a single cube to represent ones units, a fused stick in which etched lines indicate 10 single units, a fused block of ten of these ten sticks, as well as a cube with etchings intended to represent one-thousand units. In other words, these blocks by design intend to provide physical representations of multiple units at once. A single hundreds block (1 hundreds unit) is typically scored to show 100 ones unit blocks, and this scoring is done in such a way to be equivalent to 10 tens unit blocks. Although it should be noted that only the 600 squares etched on each face of a thousand cube are visible, so students typically misunderstand the intent that this cube actually contains 1000 cubes, rather than 600 (Labinowicz, 1985).

Elon Kohlberg, another professor with a PhD in mathematics, developed a commercial base-ten block manipulative called Digi-Blocks after using rocks in containers to help his nephew understand the base-ten number system (Digi-Block Inc., 2017a). The ones unit of Digi-Blocks are the only solid blocks. Each larger place value is a container that is proportional to the original unit and can hold exactly ten of them (Digi-Block Inc., 2017b), such that all the larger place values are simply *containers* or *holders* until filled with the smaller place value blocks. This means that a collection of 10 units fits inside the ten container. Then, once students collect and fill 10 ten holders, they can pack them into the hundred container and then follow the same pattern for the thousand container. Such blocks or ten frames that students can fill provide feedback signaling students when to make a new group of ten (Fraivillig, 2017). When completely separated, this tool might be considered unstructured; however, the structure of the containers requires that the only grouping that can occur is in nested sets of ten, so in effect, this tool might be considered prestructured, which they are when they are full.

All of these materials discussed thus far are considered proportional models in that an adult or child knowledgeable about base-ten structure might see or build progressively higher place value units with smaller units contained within each higher unit (Reys et al., 2014). Regardless of the type of proportional manipulative used, even if students can name the block as instructed such as "one hundred," this does not mean that the child understands or sees this block as representing a single unit of hundreds (Labinowicz, 1985). During an extended period of time, students simply see this hundred block as a convenient fusion of 100 individual blocks (Labinowicz, 1985). For as Labinowicz stated "we see what we understand" (1985, p.301).

The proportional Dienes block brand seems to be universally seen as equivalent to the generic term "base-ten block," yet Digi-Blocks are also base-ten block manipulatives. Thus, for clarity in this chapter, the term *multiunit blocks* (*MUBs*) will refer to the class of proportional blocks that include Dienes blocks, Digi-Blocks, and any other similar materials that prestructure single units and higher-order composite units.

Examples of nonproportional models are colored counters (i.e., each color represents a different unit value), coins, or abacuses (Reys et al., 2014). Mathematically proficient students and adults need to work with nonproportional models that require trading values, because they need to understand, for example, that a single hundred-dollar bill could be traded for ten 10 dollar bills. Such nonproportional models of quantities, however, cannot model the idea of groupings of groupings, composite units, or containment. It is widely accepted that such nonproportional models, which are not a focus of this chapter, are more abstract and should only come after students gain a conception of quantity through proportional models (Reys et al., 2014).

"Hands-On" Learning with MUBs

To set the stage for the investigation of "hands-on" learning experiences with the most common MUB, consider what you see when looking at Fig. 11.1 and how this is influenced by what you understand about mathematics that a novice does not. Figure 11.1a shows a Dienes block representation of 1040 on a workspace for a problem (purposefully withheld from the reader at the moment). In Fig. 11.1b, pay attention to the student's movement and what it models or represents about arithmetic. What is the students' hand doing? Do you agree that the hand removes, takes out, or takes away one-thousand cube? What does this physical movement represent arithmetically?

Physical manipulatives are a common way to support students to learn intended or targeted ideas. Such materials have been acknowledged as metaphors, microworlds, or models of abstract mathematical ideas (Nesher, 1989; Pimm, 1981). Moreover, student use of such manipulative materials has been referred to for decades as "hands-on" learning. Paradoxically, research and practice have not attended to what students' hands mathematically represent during "hands-on" learning such as the questions I raised in relation to Fig. 11.1. Thus far, research about what happens during learning experiences with such materials has focused on the visual arrangements of the blocks after students move blocks. In other words, these student movements have been treated as a necessary step to get the physical

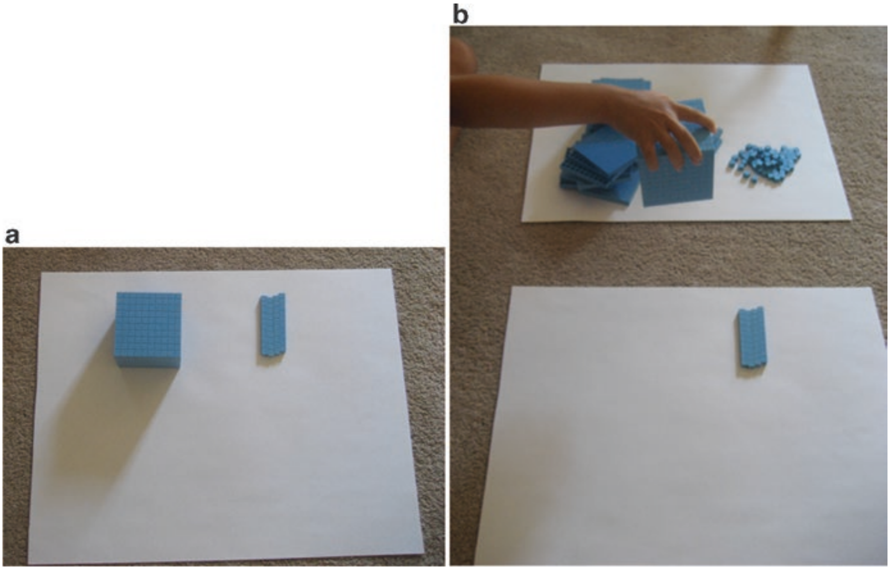

Fig. 11.1 Quantity 1040 modeled with Dienes blocks (**a**) before student's hand moves the one-thousand cube in (**b**)

arrangements to visually represent numerical quantities. However, this is a static perspective of students' entire experience. This implies students' experiences consist of a compilation of still-frame photos. Research must use a video approach (both literally and metaphorically) to view students' experiences in order to understand the actual process and potential causes of learning outcomes. To reveal what has gone unnoticed about the processes, I will focus on what happens between each resulting photo. That is, within a dynamic process of solving a problem with MUBs, I ignore the commonly portrayed resulting photos readers might expect to see in order to attend to students' physical motions that moved the blocks and what those motions (and seeing the blocks move) arithmetically represent. To provide justification as to why prior still-frame perspectives have limited the field's understanding of the learning process and why new perspectives are needed, research on how physical motions influence cognition and evoke metaphorical concepts will be discussed before examining cases of how students move particular types of MUBs.

Embodied Cognitive Perspectives

Embodied cognition encompasses a variety of research foci such as investigations of how existing ideas are grounded in prior physical experiences with the world, how real-time interactions with the world influence cognition, and how verbally expressed metaphors reveal embodied bases of cognition (Glenberg, 2010).

According to embodied cognitive perspectives, physical motions evoke concepts even if the language used does not communicate this idea (Antle, 2013; Goldin-Meadow, Cook, & Mitchell, 2009). Much research has been conducted with adults that supports the claim that the influence of human movements on thinking is more than a developmental stage of childhood. For example, Antle (2013) found that adults who watched images of humans with inequitable resources were more likely to not only notice the inequity (abstract imbalance of resources) but also express a desire to correct the imbalance, if while watching the images they had to work to keep their whole body physically balanced on a platform compared to those who rotated a joystick (Antle, 2013). In other words, the physical motions of balancing influenced people's concepts to see the same images with different meaning.

Another implication of the Antle (2013) study is evidence that the consistency between concepts and motions matters, in this case, consistency of the underlying concepts of balance and imbalance in an abstract metaphorical sense with people's physical movements. In another study the physical motion of adults moving objects from one bowl to another that was away from them or toward them evoked the underlying idea of away and toward, which affected their comprehension of literal and metaphorical written sentences (Glenberg & Kaschak, 2002).

All of these examples provide evidence that how humans move influences how they think. Moreover, these studies also evince the importance of consistency of physical motions with the intended ideas or concepts. Additional examples have emerged that show consistency of physical motions matter for third- to sixth-grade students learning mathematics (Goldin-Meadow et al., 2009; Nurnberger-Haag, 2015). That is, that students' physical motions serve a metaphorical purpose, and sometimes their motions led them to verbally express this metaphor. Specifically, in Goldin-Meadow and colleagues when the students were taught to put two fingers of one hand on the two addends of an equation that should be simplified and point one finger of the other hand to the relevant number on the other side of the equation, this evoked for the students the concept of putting together two addends (Goldin-Meadow et al., 2009). Moreover, those who were most successful articulated this metaphor verbally as "grouping." The students who were in another condition who made the same motion with irrelevant numbers did not do as well and did not verbally express this grouping idea even though the motion was the same (Goldin-Meadow et al., 2009). Thus, the relationship of the movements with mathematical objects, such as written symbols, also matters.

Such research suggests that the areas of arithmetic in which educators already use "hands-on" instructional metaphors, such as MUBs, warrant research with embodied perspectives of cognition. Analyses of how students move MUBs are needed in order to understand how their motions may be influencing their thinking and consequently their learning. The idea of grouping is a necessary but insufficient aspect of understanding base-ten arithmetic. Consequently, results about grouping numerical symbols in prior studies (e.g., Goldin-Meadow et al., 2009) suggest that analyses of the ways that students group manipulative materials would provide critical insights about student understanding of base-ten arithmetic.

First, let us consider the common ways such materials have been viewed and then show what has been previously overlooked. For example, the action of substituting blocks with the equivalent value (e.g., 1 ten block for 10 ones) is often referred to as "a trade," and the blocks that are fairly traded are usually circled within the static problem diagrams (e.g., Fuson & Briars, 1990; Labinowicz, 1985). Where these blocks that were traded came from and the four separate motions required for students to perform each trade have not been attended to in theory or in the drawn representations of the block arrangements shown in researcher nor teacher publications. Such still-frame perspectives limit what can be noticed about students physically moving tools to represent arithmetic. Just as it is already commonly understood in mathematics education that research on social interactions in classrooms must be captured with video cameras rather than photographs, investigations of "hands-on" learning experiences must also use these same methods. Given these perspectives on how movement influences cognition, what follows is the analysis of students' movements of two prestructured proportional MUB blocks (i.e., Dienes blocks and Digi-Blocks). This will be followed by a brief discussion of embodied perspectives on metaphors in relation to the instructional metaphors educators (including educational researchers) orally express.

Analysis of MUBs from Embodied Perspectives

In a different publication, I used the term *model-movements* to refer to the ways that students and educators typically move their bodies or physical materials due to the affordances and constraints of those models (Nurnberger-Haag, 2015), so this term will be used to describe movements with MUBs. For educators familiar with elementary mathematics, Dienes blocks as classroom materials may be as commonly accepted as any other tools such as chairs whose purpose and function no longer require effortful attention. Thus, in order to see something so common from a new perspective, it is often necessary to hide aspects of a context that reinforce existing understandings. Consequently, to focus on how the physical and visual experiences represent ideas, let's imagine for a moment that classroom instruction with these blocks occurred without oral language or sign language. What do students' physical model-movements represent about arithmetic? Due to their model-movements, what might we hypothesize students would verbally express if teachers did not insert their own language to this process?

MUBs that Require Trading If proportional models like Dienes blocks or non-proportional models are used, students must physically trade one place value unit for a different unit in order to calculate with these materials. Next, I trouble what these trading requirements could conceptually mean or mathematically represent in order to reveal potential reasons that such materials may fail to support student learning of conceptual structures in intended ways.

Trading Model-Movement Model-Unintended Operations Trading and equivalence are key themes in mathematics, particularly for solving equations; however, in this context of multi-digit calculation and place value, trading is an unnecessary metaphor. Trading is actually composed of physical giving and taking movements that a student needs to consider together as an abstracted fair trade. Based on research about how physical movements subconsciously influence human thinking (Glenberg & Kaschak, 2002), even if students agree that they completed a fair trade, the students' physical model-movements of putting in (add) quantities and then taking out (subtract) quantities likely activate ideas of addition and subtraction at unintended times that may interfere with learning.

Revisit Fig. 11.1, which shows a student taking a thousand block away from a representation of the quantity 1040. This model-movement could be representing the subtraction problem 1040 − 1000, because taking something away is one way to model subtraction. An educator might also recognize such movements as the first step of processes with Dienes blocks to trade 1 thousand for 10 hundreds; however, the physical model-movement to perform this first step of a trade is the same as students' movement to subtract 1000. Regardless of how an educator might intend that students see or think about the action as part of a trade, students' actual motions model taking away or removing. In other words, students' physical model-movements with Dienes blocks model subtraction operations even when unintended. In Fig. 11.1b, notice also the collection of extra blocks, which is where the student is moving the thousand cube to. This student was actually demonstrating a first step of the problem 1040–463.

Table 11.1 explains student model-movements to calculate 1040 − 463 with Dienes blocks using the trade-first left-to-right subtraction algorithm Fuson and Briars (1990) used and subsequently found in elementary textbooks (e.g., The University of Chicago School Mathematics Project, 2012). Although in practice I encourage students to use methods that make sense to them, for space and illustrative purposes, this chapter explains the problem using the particular algorithm Fuson and Briars (1990) indicated students find more beneficial than a traditional algorithm. The second column of Table 11.1 shows each action and quantity using numerical expressions to illustrate the unintended arithmetic similar to the method Vig, Murray, and Star (2014) used to illustrate how a chip model represents addition and subtraction of negative numbers.

In order to trade 10, students may not be able to instantly grab 10 and only 10 of a certain sized unit block. This means students may have more than four separate movements in order to prepare objects for trading (count out and gather each set of 10). To focus attention on how all students would move to trade, the table focuses on the four main trading actions for the sake of argument.

Note the processes needed to enact a single trading metaphor require at least four separate movements (see Table 11.1). Each of these movements is indistinguishable from how students move to represent intended operations. In the rightmost column of Table 11.1, notice that more of the student's movements represent unintended operations of both addition and subtraction in unintended situations than intended subtractions. I hypothesize that such unintended operations could interfere with elementary student learning of whole number operations as prior research had found for older students learning integer operations. For example, this interference was found with fifth- and sixth-grade students using color-coded counters to represent positive and negative numbers (Nurnberger-Haag, 2015). Students who experienced

Table 11.1 Descriptions of Dienes block model-movements to reveal unintended and intended operations, using example of 1040-463

Verbal description of movements	Numerical representation of model-movements in each space	Operation model-movement meaning	Pedagogical intent	Model-movement operation match
Take away 1 thousand cube	1040 – 1000 = 40	Subtract	First quarter of trading action	Unintended
Put thousand cube with blocks external to problem	E + 1000	Add	Second quarter of trading action	Unintended
Take 10 hundred blocks away from external blocks	(E + 1000) – 1000	Subtract	Third quarter of trading action	Unintended
Put 10 hundred blocks with problem blocks	40 + 1000 = 1040	Add	Fourth quarter of trading action	Unintended
Take 1 hundred block away from problem blocks	1040 – 100 = 940	Subtract	First quarter of trading action	Unintended
Put 1 hundred block with external blocks	E + 100	Add	Second quarter of trading action	Unintended
Take 10 ten sticks away from external blocks	(E + 100) – 100	Subtract	Third quarter of trading action	Unintended
Put 10 ten sticks with problem blocks	940 + 100 = 1040	Add	Fourth quarter of trading action	Unintended
Take away 1 ten stick from problem blocks	1040 – 10 = 1030	Subtract	First quarter of trading action	Unintended
Put 1 ten stick with external blocks	E + 10	Add	Second quarter of trading action	Unintended
Take away 10 single blocks from the external blocks	(E + 10) – 10	Subtract	Third quarter of trading action	Unintended
Put 10 single blocks with problem blocks	1030 + 10 = 1040	Add	Fourth quarter of trading action	Unintended
Take away 4 hundred blocks	1040 – 400 = 640	Subtract	Subtract	Intended

(continued)

Table 11.1 (continued)

Verbal description of movements	Numerical representation of model-movements in each space	Operation model-movement meaning	Pedagogical intent	Model-movement operation match
Take away 6 ten sticks	640 – 60 = 580	Subtract	Subtract	Intended
Take away 3 single blocks	580 – 3 = 577	Subtract	Subtract	Intended

Note: E = the unknown value of blocks represented in the extraneous or external trading zone

integer instruction with counters compared to a number line model did worse on problems for which they had to put in extra counters (addition model-movement) in order to subtract as the problem required, compared to those problems that did not require these unintended addition operations (Nurnberger-Haag, 2015). Moreover, there are at least two other related problems this trading constraint of the materials creates that have potentially unintended consequences: opening a closed system and failing to model base-ten ideas of containment.

Trading Violates Closed System If students solve a sum of 14 and 28, for example, there are many ways students could conceptually use ones units or a combination of ones and tens units as promoted with Number Talks (Parrish 2011). All of these ways of thinking allow students to think about combining the quantities 28 and 14 within a closed system of those quantities. The physical limitations of Dienes blocks or any other materials that require an exchange of ten of one thing for another pose another potential issue that may impact students' conceptual structures. These blocks require students to treat a given arithmetic task as an open system, which is inconsistent with base-ten ideas. When students use Dienes blocks, they must introduce additional blocks from outside the system of the given problem, in other words open the system to include extra blocks that do not directly model the problem. These extra blocks are irrelevant to the arithmetic problem at hand but necessitated by the particular instructional metaphors. In other words, the trading model-movements students must enact open the system to include this trading zone of extra blocks. In this way, it creates an "otherness" that is unnecessary and potentially confusing (similar to Table 11.1 columns 1 and 2). That is, students need to leave the block representation of the problem at hand to go to this other source of blocks that becomes conflated with the blocks intended to represent the problem. For example, to use Dienes blocks to calculate 28 plus 14, after collecting 2 tens, 8 ones, 1 ten, and 4 ones or 42 total ones, students temporarily reduce the quantity modeled in the problem space from 42 to 32. Students remove ten of these ones from the quantity being considered and go to a trading pool of blocks external to the quantity to get this "other" ten to exchange. This means a student works with a total unintended quantity of 52 ones during the course of solving the intended problem (42 from the original system and 10 additional from the external stash of blocks). Research on consistency of movements with cognition (Glenberg & Kaschak, 2002; Goldin-Meadow et al., 2009) would suggest that it may be counterproductive for students to imply that one has to externally trade some other values with the values in the problem to which we want students to attend. Do students mathematically categorize in their mind and distinguish between the blocks meant to represent the problem and the extra blocks that serve only as a repository to make trading blocks possible? From psychological perspectives, is the additional cognitive load useful or a source of interference?

In contrast, the Digi-Blocks support students to combine 28 blocks (which they could represent as two containers of 10 blocks and 8 additional blocks) with one container of 10 blocks and 4 additional blocks. The 42 total blocks remain together as part of a closed system. The only external objects students bring to the problem system are containers to organize or structure the single quantities into units of ten. Thus, these containers serve an organizing function, not a block in of itself. The higher-level unit of tens does not exist without the basic level unit; adding a container is not the same as adding the thing it holds. This differs from changing the number of blocks in the problem system in the ways Dienes blocks require. Such comparisons of affordances and constraints of these various materials warrant investigation for the potential intended learning and, with respect to Dienes blocks, unintended interferences of learning.

Trading Fails to Connote Containment At least one other unintended consequence of proportional materials such as Dienes blocks that require opening what should be a closed system is that they fail to model the successive containment of units of the base-ten number system. It is crucial that students develop understanding of the same quantity in terms of different sized units (Steffe & Cobb, 1988). In regard to linear or other forms of measurement, it is more productive for a person to understand that 1 kilometer contains 1000 meters than needing to think that 1 kilometer must be traded for 1000 meters. This idea is equally important for thinking about base-ten number units. The learning objective is not for students to think that 1 one-thousand unit must be traded for 1000 ones or 10 hundreds units rather that each unit contains those values. Some MUBs can model this containment idea such as Digi-Blocks, although modern Dienes blocks do not. So if a sum of ones were 42 units, for example, mathematically knowledgeable people can see this same quantity of 42 using different ways of categorizing or decomposing the units: 42 individual ones, containing 4 tens and 2 ones, 3 tens and 12 ones, and many other ways.

Theoretically, metaphors that support ideas of containment should better support student learning, because ideas of containment reflect the intended mathematical ideas in ways that also build on innate cognitive mechanisms. Research on how people learn and think about categories has identified base level and superordinate as well as subordinate categories (Rosch et al., 1976). Mathematically, place value units are categories with a base level (ones units), superordinate levels (tens, hundreds, etc.), and subordinate levels (tenths, hundreds, etc.). Research supports the claim that humans think of categories metaphorically as though they are containers (Boot & Pecher, 2011; Johnson, 1987). Consequently, it is important to consider how metaphors could influence learning of a category-based topic such as base-ten arithmetic structure. This should lead us to test how materials that encourage students to move in ways that put in and remove objects from containers or physically build structures of contained or nested quantities might afford building conceptual understandings of category units. Thus far I identified the ways students move and see physical materials used for base-ten number that do and do not support these ideas as well as metaphors educators orally express. These theoretical analyses warrant studies that use a range of methods from psychology-based experiments to investigations of classroom-based instruction.

Deconstructing Orally Expressed Instructional Metaphors

In the previous section, I asked the reader to suspend knowledge of the reality of classrooms to ignore student and educator use of language in order to focus on what students' movements would physically represent (i.e., metaphors they might physically enact with various tools). Now consider the real classrooms in which teachers and textbooks use language to explain what they intend students perceive. Consider whether and how the metaphorical meaning students may experience by physically moving those tools relate to the following discussion of the terms textbooks and educators use orally and aurally. Educators (including educational researchers) have recognized and discussed the use of analogy and metaphor to teach content areas including mathematics (English, 2013; Pimm, 1981). Yet, to my knowledge, the particular terms for base-ten arithmetic have not been discussed as metaphors in prior work, so I analyze them here in terms of their intended and unintended mappings to addition and subtraction operations. Several terms have been used such as *carry*, *borrow*, *trade*, *group*, *ungroup*, *regroup, pack*, and *unpack* (Digi-Block Inc., 2017c; My Math, 2013; SRA Concepts, 2013; The University of Chicago School Mathematics Project, 2012). Elsewhere detailed mappings will illustrate how each of these terms maps from source to target in intended and unintended ways. Due to space and for clarity of the general framing of this chapter on instructional metaphors, the following focuses on revealing the primary issues with such metaphors.

Carry and Borrow When people use the terms *carry* and *borrow* in the context of addition and subtraction calculations, unlike the rest of the terms analyzed here, people may not think of the typical meanings of the terms *carry* or *borrow*. That is, due to the specific mathematical context in which adults previously practiced the terms, when adding, they may associate *carry* as meaning literally to inscribe a 1 or 2 as needed to the left of a place value or, when subtracting, *borrow* as meaning to cross out, reduce, and place a 1 next to the ones digit. If, when using these terms in this context, people only associate it with that particular meaning instead of other or original metaphorical meanings of the terms *carry* and *borrow*, then these would be considered "dead metaphors" (Lakoff & Johnson, 2003/1980). In other words, what began as metaphors to facilitate understanding between adults of a known idea to an unfamiliar idea adults now think of as having a literal meaning (Lakoff & Johnson, 2003/1980). Due to these extensive experiences, even if adults conceive of these terms as names for literal algorithmic procedures, whether students novice to base-ten arithmetic expect these *carry* and *borrow* terms to help them learn arithmetic using the meanings they already understand warrants research. Consequently, this theoretical analysis deconstructs the meaning of *carry* and *borrow* to trouble these instructional metaphors.

Carry means "to transfer from one place (as a column) to another" (Carry, 2017). The meaning of carry implies that the position of the same item is simply transferred or moved. Yet, *carry* fails to connote the intended mathematical idea that students should conceive of a carried value as a different unit. In the case of adding, for example, a written notation of a 1 may be procedurally transferred; however, once moved, it becomes ten times the value.

Moreover, the term *borrow* fails to reflect conceptual meaning and procedures. Borrow means "to receive with the implied or expressed intention of returning the same or an equivalent" (Borrow, 2017). Thus, this term is a misnomer because when teachers or other adults say "borrow from the tens place," for example, there is never an intention of returning the equivalent value of ones back to tens. The term or phrase *gifting* or *taking* might more accurately reflect this written mathematical procedure. Yet none of these terms reflect the intended mathematical ideas or procedures of converting a large composite unit into ten times the next smaller unit.

Furthermore, the pair of terms *carry* and *borrow* are meant to represent processes for addition and subtraction, respectively. Given that addition and subtraction are inverse operations, an effective instructional metaphorical pair would likely communicate the inverse relationship. However, using the definitions above, it is clear that the term to *borrow* is not the inverse of to *carry*.

Trade Given that the terms *carry* and *borrow* were in use long before the term *trade* became part of school mathematics, were it not because of the popularity of Dienes blocks and some research using these terms because of these blocks (e.g., Fuson & Briars, 1990), then we should have seen the term arise much earlier. For as Labinowicz (1985) explained, with prestructured materials such as Dienes blocks, students can only decompose blocks "indirectly by trading" (p. 273). This term "trade" like "bundle" referred to earlier has been treated as though it describes or is the conceptual and literal meaning of an arithmetic process (e.g., Fuson, 1990; Saxton and Cakir 2006), which this analysis aims to reveal is really an instructional metaphor that fails to reflect the processes.

Although the physical actions the verbal metaphor trade implies are consistent with how to physically use the Dienes block material, this term is inconsistent with ideas of base-ten numbers or even written procedures. Even if students do not use the physical Dienes blocks, if an educator were to use the term "trade" verbally with written symbols, it is important to consider the limitations of this verbal metaphor. The idea of trading one set of values for another is crucial in mathematics; however, the term trade fails to communicate the hierarchical structure or nesting of units and composite units. Consequently, next let's deconstruct the meaning of this verbal metaphor.

The vernacular term *trade* means "the act of exchanging one thing for another" (Trade, 2017). This idea of trading implies some degree of perceived equivalence, in that children, for example, might trade different numbers of valued treasures based on their perceived values (e.g., three trading cards for one necklace). The term *trading*, however, gives no indication of a change in level of these units, which is an essential characteristic of the base-ten number system. An educator could encourage students to articulate the units they are trading to compensate for this limitation of the term (e.g., trade 10 ones for 1 ten). Yet, consider that even for the written procedures, moving 10 ones in the ones place to 1 ten in the tens place does not convey the meaning of trade in either the childhood or commercial sense. In order for arithmetic operations to reflect the denoted meaning of *trade*, the ones and tens values would need to switch places. An exchange or transaction in life means each person has something different than before, which does not occur arithmetically. In the algorithm, the reason 10 ones are changed into 1 tens unit is because a ten contains 10 ones.

Educators use the term trade to refer to both directions of processes, meaning the term does not reveal if one is converting a unit into the next higher-level or lower-level unit. The verbal metaphor fails to represent the direction of the intended action and thus fails to reflect the inverse nature of addition and subtraction operations.

Grouping Metaphors One basis of the Hindu-Arabic number system is grouping by ten. Thus, terms related to the idea of grouping might seem to be productive verbal metaphors to communicate base-ten number structure. Unlike the other terms discussed here that have single forms, educators use multiple variations of the term *group* as metaphors for the arithmetic processes: group, regroup, and ungroup. Let us compare each of the terms used in practice to how they may or may not facilitate the base-ten number structure with various manipulatives and then summarize these as related collection of terms.

Group The meaning of "to group" that would be most common for students would be "to combine in a group" (Group, 2017). Although putting objects together into groups of ten is necessary to build base-ten structure, this is insufficient. Successive groupings of those groupings are required (Labinowicz, 1985).

Regroup Decades ago, standards and textbooks classified problems as addition or subtraction with and without borrowing or carrying and then shifted to classifying such problems simply with the new term regrouping, as in "the student will subtract two-digit numbers with regrouping." Given the critiques of the terms *carry* and *borrow* shared earlier, the change to a "grouping" metaphor may more accurately represent the underlying arithmetic ideas, but let us deconstruct the term *regroup*. The prefix "re" means "again." Thus, regroup means "to form into a group again" or in practice "to form into a new grouping" (Group, 2017). This term could represent well the mathematical actions of regrouping a quantity such as 8 into 4 and 4 and then 3 and 5. Similarly, the quantity 14 can be grouped as 7 and 7 for a doubles strategy or 10 ones and 4 more ones. These examples, I argue, reflect meanings of the term regroup that are consistent with mathematical ideas. These groupings, however, are different arrangements within the same unit size or level. The idea of regrouping or to form into a new grouping fails to connote constructing superordinate or subordinate units. When students obtain 10 groups of tens either strictly with written symbols in an algorithm, with popsicle stick bundles, or some other materials such as Dienes blocks, some textbooks tell them they need to "regroup" into one hundred (e.g., My Math, 2013). This could simply mean to change the group size for efficiency as when counting by twos or fives, so this term "regroup" may not support the intended learning goal of reorganizing student thinking to a higher-order unit.

Another issue with the term *regroup* is that it is used for both addition and subtraction. Thus, *to regroup* does not indicate to students whether to make a quantity into a larger or smaller unit. Consequently, it cannot support the idea of inverse operations.

Ungroup The terms group and ungroup when used together could convey the inverse nature of how to move materials such as straws, sticks, and individual blocks to do and undo or put together and take apart. In other words, the pair of terms *group* and *ungroup* could connote the inverse operations of addition and subtraction.

Summaries of Grouping Metaphors All of these variations of groupings could support expanded notation algorithms or student thinking and invented strategies about individual units or ones. Consequently, this may be a useful initial verbal metaphor. A related phrase that may better, albeit awkwardly, describe the hierarchical structure of base-ten number system would be "groups of groups" (Labinowicz, 1985, p.273). Yet, in practice, such uses seem to be rare; instead educators who express metaphors of grouping use terms that reflect a single-level unit or moving from one type of grouping to another, rather than the building of higher-order units.

Pack and Unpack Some classic problems, such as the candy-packing problem in which students are given a task to pack candy into boxes that hold ten candies and then into shipping boxes that hold ten of each box (Heuser, 2005), have been used in practice and in research. The terms *pack* or *unpack,* however, seem to have been used only when this literal meaning of packing motions applied to the problem context. Yet, the term *pack* may be a potentially powerful verbal metaphor for abstract base-ten number structure, because it could promote the idea of units contained within other units. When researchers such as Kamii have provided diagrams to encourage researchers and educators to think of individual units as a collection, they draw a loop around the collection of individual units to refer to that ring as the new collection or container (Kamii, 1986). Although such researchers have not invoked the term packing in these scenarios, the ideas are about *containment*. Consider that the term *pack* at a minimum implies the idea of multiple objects contained within some other type of object that serves as a container. Once a student has 20 ones (single objects), for example, the student has two full containers of 10 ones, the containers of which are the composite unit "2 tens."

Consequently, I claim that the term pack is not a synonym for other terms used as instructional metaphors for base-ten number. Notice that these definitions of *pack* and *unpack* refer to multiple levels of objects at once. These are the objects and a unit that contains those objects (container). Other terms such as *borrow*, *trade*, or *group* connote only working within the same level categories, so they do not communicate the idea or need for higher-level units or superordinate categories. At least in theory, the terms *pack* and *unpack* better reflect this nested unit structure of the base-ten number system.

Definitions of *to pack* include “to fill completely” and “to put items into a container” (Pack, 2017a, 2017b). This meaning of *pack* can serve as an instructional metaphor for teaching base-ten structure using Digi-Blocks (Digi-Block Inc., 2017c). Consider arithmetically that ones units cannot form “a ten” until the idea of a ten unit is completely filled. The number system involves this ten structure that students must learn when and how to fill or pack each successively higher unit if and only if completely filled. The container represents the idea of a different unit.

The definition of the term *unpack* makes even more explicit the need for a container: “to remove the contents of” or “to remove or undo from packing or a container” (Unpack, 2017). Such an analysis of the instructional metaphors *pack* and *unpack* opens many questions for future research. For example, how might verbally describing arithmetic processes with the pedagogical metaphors *pack* and *unpack* support students to think about each place value as being contained within successively larger place values by a factor of 10, irrespective of whether students physically pack objects to model quantities?

Conclusions

Intended and unintended meanings of many common instructional metaphors for base-ten arithmetic have been analyzed in this chapter, both those that might be evoked through students’ physical motions and those that educators verbally express. The following concludes by summarizing the single metaphors analyzed here as a hypothetical exercise and then discusses potential issues with mixing these metaphors, which reflects potential issues of real classroom instruction.

Single Metaphors

The metaphors discussed are all tools used with the intent to facilitate students’ conceptual and procedural development of base-ten number. Regardless of the form in which the metaphors might be evoked, whether verbal, visual, or physically enacted, some metaphors insufficiently map to the targeted base-ten number structure, whereas others contradicted or were inconsistent with this structure. Thus, most of these tools may be ineffective for the intended job. *Group*, *regroup*, and *ungroup* are in theory insufficient metaphors in that they addressed part but not all of the essential ideas of base-ten numbers. Whereas, materials that promote physical motions or oral terms such as *carry, borrow*, or *trade* promote several unintended meanings that are inconsistent with what educators intend students learn.

In particular, the pervasive *trade* metaphor may serve the unintended function of the knife in the quote that began this chapter. Even if educators avoid trade as a verbal metaphor in favor of a variation of the term *group*, the materials educators provide such as Dienes blocks would still encourage students to experience arithmetic by physically enacting a trading metaphor. Whether verbal metaphors or enacted model-movements, trading violates the intended mathematical ideas and procedures, potentially distracting, interrupting, or causing inconsistencies when students experience these metaphors during instruction. This analysis revealed that one primary issue is that when students trade blocks for multi-digit calculations to model the intended operation (e.g., taking away blocks to model subtraction problems), their model-movements actually represent a greater number of contradictory addition and subtraction operations. These unintended inconsistencies between the model-movements and mathematics may interfere with learning base-ten number, because Nurnberger-Haag (2015) empirically found the same interference when students learning integer operations with chips had to put in or add chips when such addition operations were unintended operations.

The theoretical analysis in this chapter suggests that empirical investigation is needed to test the assertions that the materials that would encourage physical model-movements most consistent with the targeted mathematical ideas are materials that afford packing and unpacking groups of groups of ten and verbal metaphors that reflect these packing model-movements. For decades, methods textbooks for elementary mathematics have mentioned packing objects (Reys et al., 2014; Van de Walle et al., 2010), but aside from Labinowicz (1985) who recommended that grouping objects should come before Dienes blocks, such approaches were suggested simply as one of many potential groupable manipulatives that educators could offer students. This is understandable since the physical motions students enact to use these materials had largely been ignored, which this analysis used embodied cognition to reveal. The enacted and verbal metaphors pack/unpack reflect inverse operations consistent with addition and subtraction. Materials that encourage packing and the verbal terms pack/unpack maintain a closed system that reflects containment of multiple unit levels (i.e., within the given quantity of the problem, there are enough hundreds, tens, or ones to subtract or add whatever is needed without opening the system to an external source of blocks to find these sufficient quantities). Moreover, when students take away or put in blocks with these tools that promote a packing metaphor, each student motion represents intended arithmetic operations. Thus, bringing embodied cognition and other disciplinary perspectives to bear on the problem of how typical tools foster students' base-ten number understanding and how to design and choose better tools could help the field notice when pedagogical practices cut like a knife, in favor of tools that better serve the intended job.

Limited Metaphors Limit Conceptual Categories

Lest someone might argue that the limitations of verbal metaphors described here may not be crucial, consider that research from embodied perspectives has shown that oral terms that have a basis in prior physical motions prime those same ideas by neurally reactivating much of the pathways of those movements (see Kontra, Goldin-Meadow, & Beilock, 2012). Moreover, evidence from cognitive research that does not draw on embodied perspectives has shown that terms adults use influence both what children notice and do not notice in environmental stimuli leading to changes in how children categorize concepts (Plunkett, Hu, & Cohen, 2008). Such evidence indicates that the words educators use for base-ten number would likely influence how students' concepts are structured. Consequently, research that investigates these nuances of verbal instructional metaphors is warranted.

Mixing Metaphors

A single metaphor or representation will provide certain information and lack others (Johnson, 1987/1990). The response to limitations of representations in mathematics education has been to promote multiple representations as beneficial for learning (Goldin, 2003). In the United States, the use of multiple models is encouraged rather than making sure that students have a deep understanding of a single model, which should lead us to recognize that metaphors may be mixed. An example of mixing metaphors during instruction could be an educator who verbally expresses a grouping metaphor yet encourages students to physically enact a trading metaphor with Dienes blocks. Investigations are needed to test intended and unintended outcomes of mixing metaphors during instruction. There is evidence that mixing valid but incongruent metaphors interferes with comprehension of concepts even when adults already understood each metaphor and the target concepts (Gentner, Bowdle, Wolff, & Boronat, 2001). Consequently, how mixing metaphors influences children's thinking when learning and developing complex concepts of base-ten number structure is a crucial understanding for the field to investigate. Although multiple metaphors may be needed over time, because no metaphor can fully convey targeted ideas, questions for research include which metaphors should be used, in what ways, in what sequences, and how to connect these meanings for robust concept development.

Call for Transdisciplinary Research

Research that transcends disciplinary boundaries is needed to understand the effects of single instructional metaphors used for base-ten arithmetic as well as how mixing particular metaphors influence student experiences and learning. One approach could be for multiple studies each from divergent individual disciplinary perspectives to be conducted and encourage researchers across disciplinary boundaries to learn from and compile this collective knowledge rather than citing primarily within particular disciplines. Moreover, studies that merge perspectives within individual designs could be conducted to reflect transdisciplinary contributions to apply their current perspectives to the study of this problem of how to help elementary students develop understanding of base-ten number structure and operations.

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