Engaging Mathematically in Synchronous Platforms: Examples and Insights

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ABSTRACT

In a qualitative self-study, two teacher educators introduce the notion of *engaging mathematically* to study synchronous interactions in two of their online courses for K-8 teachers. By studying the interactions between themselves and their teachers, the teacher educators are able to describe novel opportunities, negotiations, struggles and insights involved in engaging mathematically in online platforms. Their mathematical and pedagogical illustrations convey new possibilities for synchronous online interactions during mathematics lessons. These descriptions address a gap in the research on online teaching about how mathematics can be negotiated within these platforms, as well as concerns about the meaningfulness of interactions in online settings. Implications to teacher education practitioners and researchers, and developers of learning management systems suggest the importance of the teacher education community taking a lead role in ensuring that online teaching has a purposeful part to play in the field of mathematics teacher education.

INTRODUCTION

The current study stems from the two authors' attempts to support teacher enrollment and participation in one of their graduate level, teacher education programs. The program is intended to help practicing middle school teachers, or teachers certified in another area, to become certified to teach mathematics in middle schools. Because a large portion of our audience consists of practicing teachers, part of our efforts for addressing enrollment went to transitioning the program's five content courses to an online format. The current article focuses on one of our experiences teaching two of the program's courses in this new platform.

As seasoned mathematics and teacher educators, we have always believed in the importance of social interactions to our students' learning of mathematics. This becomes especially important when our students are *teachers* who are watching us teach and learning how to teach through these observations (Feiman-Nemser, 2012; Lortie, 1975). As we transitioned our program, we realized that one of our biggest challenges concerned how to create opportunities for the kinds of interactions that we valued in our face-to-face classrooms. Would we be able to promote interactive activities, and support the discussions that surround them, in online platforms? In this paper, we describe our attempts to promote and support such interactions in one *synchronous* online platform. Our objective in this study is to provide an "existence proof" within teacher education that synchronous interactions *can* promote problem solving and discovery, and to complement these descriptions with our reflections. We were guided by the question, "What can synchronous teaching in mathematics teacher education look like, and how does a teacher educator negotiate interactions in this new environment to support a given philosophy of learning and teaching?" In our narratives, we illustrate and implement new pedagogies and negotiations, and discuss the kinds of opportunities and challenges afforded within one synchronous platform for an audience that is wondering about incorporating synchronous learning into its teacher education programs.

FRAMEWORK

From Face-to-Face to Online Interactions: Engaging Mathematically

We position ourselves amongst mathematics educators who believe in the value of social interaction to the meaningful learning of mathematics (Borasi, 1994; Cobb, Wood, Yackel, & McNeal, 1992; Confrey, 1990; Kamii, 2000; Lampert, 1989, 1990; NCTM, 1991, 2000, 2014; Piaget, 1947/1963; Richards, 1991). These beliefs originate from the principles of *constructivism* which recognize that knowledge is not passively received, but actively constructed, by the learner (Noddings, 1990; von Glasersfeld, 1990). Part of this construction recognizes how the experience of interacting with others can stimulate learning because it obliges us to consider our thinking from the perspective of others (Kamii, 2000; NCTM, 2000; Piaget, 1947/1963). The mathematics education literature has provided valuable examples of how interactions afford opportunities to learn during periods of conflict and confusion (Borasi, 1994); during negotiations of new norms for classroom interactions (Lampert, 1990; Wood, Cobb, & Yackel, 1991); and during investigations of students' problem-solving tangents (even those unfamiliar to the teacher) (Fernández, 2007). Within these studies, we noted the conversational activities of *publicly* questioning, guessing, disagreeing, struggling, negotiating, explaining, or justifying, and how these can invite students and teachers into becoming a community of mathematical inquirers who are focused on the *process* of *doing* mathematics versus the search for a right answer (Duckworth, 2006; Lampert, 1991; Richards, 1991).

In the context of online teaching, "interaction" takes on new meanings. One common differentiating lens is a temporal one, which distinguishes asynchronous from synchronous interactions. *Asynchronous interactions* provide an opportunity for participants to interact with each other, possibly at different times, using a technology forum like a discussion board. *Synchronous interactions* also are mediated via a technology forum, but participants are present at the same time, just not necessarily in the same place. The research on online teaching concerns itself with each of these settings and with student populations that can include teachers in the early or professional stages of their education. For example, experimenting with discussion boards has been shown to help teachers interact with each other about topics like student thinking (Klein, Fukawa-Connelly, & Silverman, 2016/2017; Spitzer & Phelps-Gregory, 2018). And in an online geometry class for in-service teachers, a majority of respondents indicated their support for synchronous interactions during whole-class or small group sessions (Ku, Akarasriworn, Rice, Glassmeyer, & Mendoza, 2011).

In the current study, we focus on the *synchronous* dimension of online teaching, in part, because this category has been shown to be more useful for promoting social interaction than the asynchronous one (Tallent-Runnels et al., 2006). But we also note the literature's overall concern for a loss of real-time interactions within this new platform (Tallent-Runnels et al., 2006; Trenholm, Alcock, & Robinson, 2016; Swan, 2002). Personally, we gravitated toward synchronous interactions because they more closely mirror the experiences we strove to provide our own students in face-to-face environments. And we realized that a study of our efforts to uphold such experiences in online settings would afford opportunities to investigate the very issues that mathematics educators have celebrated in their research on interactions.

The content of mathematics also figured prominently in framing our work, given that much of the research in online teaching does not focus on specific disciplines, and that in the discipline of mathematics, "systematic research pertinent to the shift toward fully online instruction is rare" (Trenholm et al., 2016, p. 148). In reviewing the literature in online teaching, we perceived a paucity of research that describes how mathematics is used and negotiated in online platforms (eg., Klein et al., 2016/2017; Ku et al., 2011; Rosa & Lerman, 2011; Spitzer & Phelps-Gregory, 2018). We noted that data in current studies tend to focus on verbal interview responses or statistical measures stemming from Likert scales or counts. However, the content (or mathematics) of the courses does not figure prominently in characterizing the online experiences. Thus, we strove to bring the use of *mathematics*, its numerical, algebraic and geometric representations, and the negotiation of mathematical reasoning in online platforms, to the foreground of our study.

In order to connect the issues highlighted in face-to-face studies with our foray into online teaching, we introduce the notion of *engaging mathematically*. We say a teacher and her students are *engaging mathematically* when their investigations into a mathematics problem are not prescriptive or their outcomes are unknown. This inconclusiveness can be indicated by a teacher's or students' expressed perplexity or astonishment during the process of their investigation. Engaging mathematically also will be characterized by the teacher supporting students in exploring the problem situation, rather than dismissing, overlooking or immediately resolving it. We view the study of these situations as providing opportunities to examine the very "questioning, guessing, disagreeing, struggling, negotiating, explaining, or justifying" that can support communities of mathematical inquirers in new, online settings.

METHODS

This paper's two authors are experienced teacher educators working at Montclair State University in New Jersey. One of our programs is entitled the Teaching Middle Grades Mathematics Certificate Program. This program consists of five content courses that can be taken by participating teachers to enhance their knowledge of the mathematics and pedagogy needed to teach current middle school curriculum or to enrich elementary curriculum. As part of a larger effort to make our courses more accessible to working teachers, we initiated a transition of these five courses from their face-to-face settings to online venues. We each took one course for the 2018-2019 school year: Geometry for Middle-Grade Math Teachers (taught by Fernández in the Fall, 2018) and Measurement in the Middle Grades (taught by Leszczyński in the Spring, 2019).

To provide one setting for encouraging synchronous interactions, we employed our university's *synchronous* communication platform called *Conferences*, which is embedded within the Canvas Learning Management System (https://www.canvaslms.com). We trained in the use of Conferences with the university's support team throughout the spring and summer of 2018. After this training, we opted to design a typical Conferences screen to display three sections: the workspace, the chat window and the roster window (see Figure 1).

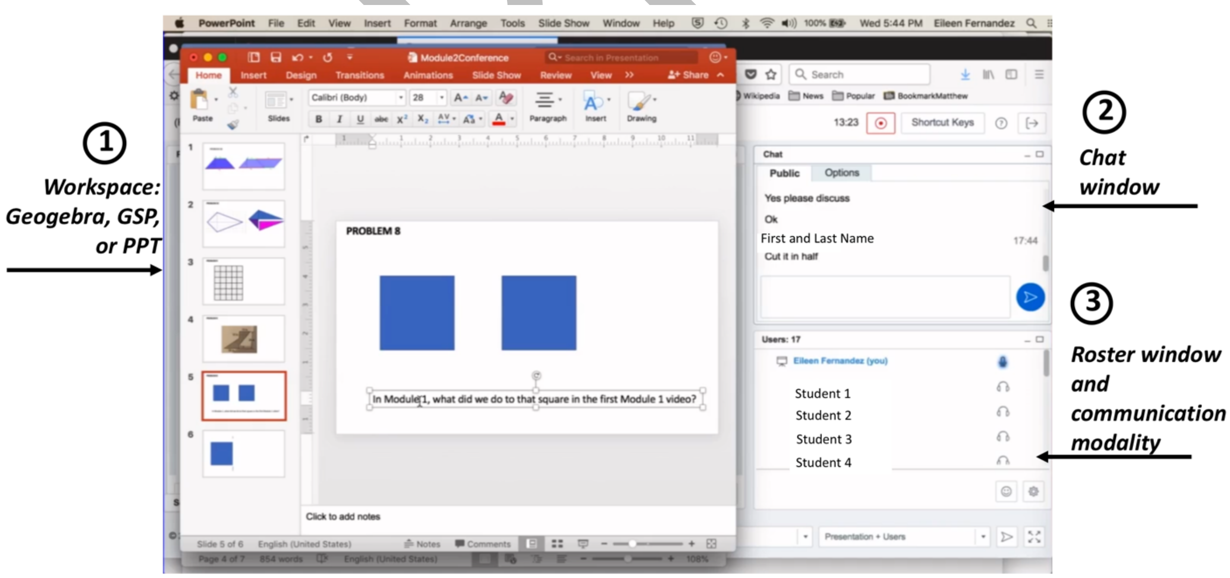


Figure 1. Typical Conferences Screen

In the *workspace*, we would share different applications with the teachers (like PowerPoint or Geogebra). For the reader, we note that Conferences restricts operating privileges solely to the instructor. The *roster window* displayed the participants' names and their preferred mode of communication (chat window or microphone or both), and the *chat window* provided a space for sharing written insights with the group. We each used Conferences in our respective courses to hold a scheduled, virtual meeting time of 2.5 hours per week. We recorded a combined 22 lessons for the benefit of participating teachers who could not make it to class or who wanted to review the lesson as support for their studies. We also anticipated using the recordings to help us reflect on our teaching.

We view our study as falling within the research tradition of a *self-study* (Feldman, Paugh & Mills, 2004). Self-study is a form of inquiry that teacher educators can bring to bear on their practice with the goal of examining, and revealing, some aspect of that practice as a resource for the teacher education community. For us, the focus for our self-study stemmed from our transition to an online format, and our concerns that participating teachers' opportunities for meaningful (synchronous) interactions could be compromised by these new arrangements (see Swan, 2002). In particular, we wanted to document how we constructed new selves, using online tools, in the service of upholding values and philosophies we had developed in face-to-face settings within this new platform. We were drawn to collaborate because of our underlying commitment to upholding constructivist principles, and because of the overwhelming nature of the challenges before us. We also were eager "to go beyond individual descriptions of changes, to develop a more complex understanding" (Feldman, Paugh, & Mills, 2004, p. 961) of the re-interpretations these transitions brought to our teaching.

Although our approach is self-study, we still need to focus on that part of our transition to online settings that would be analyzed (Patton, 2015). As noted above, our experiences within the synchronous interactions would constitute that focus. To that end, we drew from three data sources which we viewed as providing qualitative descriptors of these experiences: (1) lesson recordings, (2) our individual reflections on these recordings, and (3) our collaborative reflections. The first source, lesson recordings, provided data for online interactions. These data included spoken and written words, and workspace images where mathematical representations were displayed and negotiated between participants and managed by the teacher educator. For the second source, we listened to and studied our respective recordings, noted and elaborated examples of engaging mathematically, and each kept individual journals of our findings. We were guided by our characterization of mathematical investigations that are not prescriptive or whose outcomes are unknown. These instances intersected with those in which "questioning, guessing, disagreeing, struggling, negotiating, explaining, or justifying" arose during the course of online teaching and together with our characterization for engaging mathematically, constituted, for us, *meaningful interactions* in our synchronous settings.

Once these instances were identified, we listened to and watched each other's respective recordings, and reflected with each other about our negotiations, struggles and insights, both mathematical and pedagogical. These conversations also were recorded and generated a third source of data. This work led to our creating case studies (Creswell, 2007) of engaging mathematically using all three data sources. That is, we wrote a detailed and rich description of each case or episode of engaging mathematically and our online negotiations and strategies. To define the boundaries for our cases, we identified the posing and resolution of the mathematics problem in the lesson recording. This helped us to maintain and use the surrounding context of our mathematics problems in our analyses. Our focus on interactions drew us to analyze interactions between the teacher educator and teachers or amongst the teachers. That is, we considered how participants responded to one another during the course of investigating a mathematics problem and how lesson tools were used in these responses. We also opted to analyze multiple case studies (versus one) to provide opportunities for comparisons and contrasts (Patton, 2015). Thus, we hoped the use of multiple examples would allow for within- and across-case analysis and that these combined resources would provide a rich portrait of the individual and collaborative components of our thinking for our self-study. We used rich description to report our endeavors and insights with the goal of providing a portrait of our thinking as we attempted to engage mathematically with our teachers within this new online environment.

Ultimately, we selected five episodes to present in the current study. This selection was guided by our research question and the desire to add to that body of literature that conveys the *promise* of online interactions (eg., Davidson-Shivers, Tanner, & Mullenberg, 2000; Fernández, McManus, & Platt, 2017; Klein et al. 2017; Spitzer & Phelps-Gregory, 2018) versus the limitations (eg., Kanuka & Anderson, 1998; O'Dwyer, Carey, & Kleiman, 2007; Thomas, 2002; Trenholm et al., 2016). We also considered how our negotiations could be presented so that the reader was eased into the complexity of online interactions, and supported in studying and appreciating them. In our Results section, we present each episode in turn, the mathematics problem under study, the interactions surrounding the mathematical investigations, and our discussion relaying our negotiations and their relevance to the literature on online or mathematics education.

RESULTS

An Introduction to the Chat Window

We begin by introducing the reader to an interaction vehicle that was new to us in teaching, namely, the *chat window*. The upcoming interaction is characteristic of how the chat window enables the teachers to simultaneously articulate their thinking while the teacher educator navigates the window. Unless otherwise indicated, we refer to the "instructor" as the teacher educator (TE) and to the "students" as teachers. Pseudonyms for the teachers are used throughout our paper.

In this 4-minute long episode from the geometry class, the general topic under study is the *xy*- plane. Eager for the teachers to use features of the plane in reasoning about a problem, the TE asks the teachers to describe similarities and differences between the points *A*(3,0) and *B*(0,3). After a 27-second silence, the teachers' written observations come "scrolling" through the chat window display and the TE scrolls up and down to read and discuss them (see Figure 2).

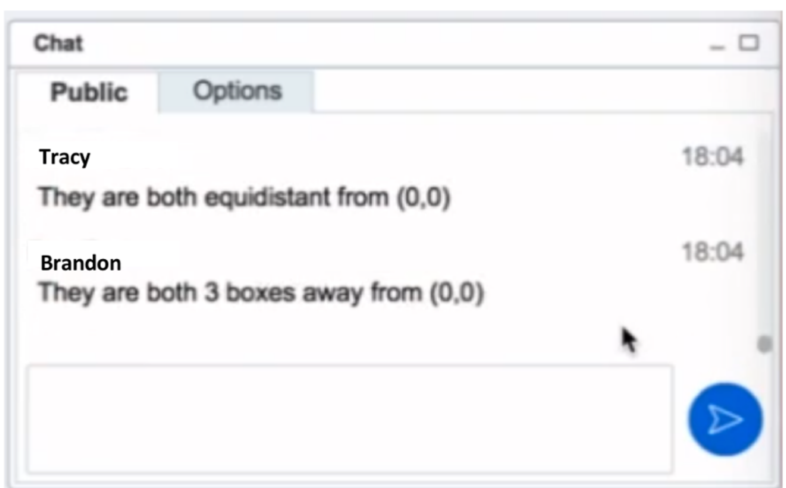


Figure 2. Similarities Between (3,0) and (0,3)

Because the text feeds come in so quickly, the TE sometimes scrolls up (to read those observations she misses) or down (when new observations are entered). The TE re-reads text entries aloud and celebrates the teachers' contributions. All told, teachers type eight observations, five similarities between the points (eg., "They both move 3 in some direction") and three differences (eg., "B moved up 3 from 0 while A moved 3 to the right").

The TE recalls this discussion being relatively easy to navigate. In part, this was because it was not necessary to sequence the teachers' entries in a particular order to create and develop the episode's arguments. She recalled her delight at unexpected observations [eg., "Neither (of the points) lie within a quadrant"] and how certain contributions reinforced other teachers' thinking. For example, in Figure 2, Tracy and Brandon's observations appear within seconds, but are reformulations of a similar phenomenon.

In reflecting on the episode, the TEs felt the chat window had enabled the teachers to use the features of the coordinate plane to communicate their thinking about the posed task. We began to see evidence that the chat window *can* promote interaction (Cook, Annetta, Dickerson, & Minogue, 2011), particularly in the service of sharing mathematical ideas. In contrast to face-to-face settings where hand-raising and turn-taking norms can determine who speaks when (Cazden, 1988), the teachers are using the chat window to share their thinking in *their* time. For us, the resulting interactions facilitated the participation of multiple contributors and the communication of multiple perspectives—each a valued trait in the discourse of mathematics classrooms (NCTM, 1991).

Negotiating Environments: Technologies and Physical Spaces

In this section, we discuss two episodes in which the TEs engaged mathematically with their teachers by integrating more familiar technologies (like PowerPoint and Geogebra) with newer ones like the chat window. The TEs' strategies and negotiations also are illustrated as they connect some of these technologies to the physical spaces in which the teachers were situated in order to engage mathematically.

Two Squares-One Triangle

In this episode, the teachers inquired about a homework problem that had been assigned on a lesson about area: *suppose you are given two squares, each having an area of 1 square unit. Use the squares to create one triangle with an area of 2 square units*. The TE worked from an un-played PowerPoint slide, that is, the slide was in Normal View, and displayed on Conferences screen-share window (see Figure 1).

In order to initiate a discussion about the homework struggle, the TE reminded the teachers to take out their paper squares and scissors at home to work on the exercise. She also reminded them about prior exercises in which they had cut shapes iteratively with scissors and investigated relationships between area and perimeter. This reminder prompted the teachers to suggest places to "cut" the two squares into triangles, which the TE diagrammed using PowerPoint's shape feature (see Figure 3a).

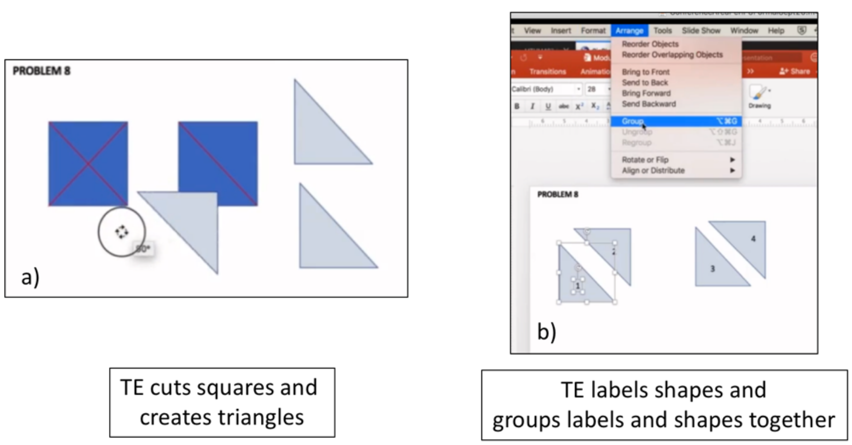


Figure 3. TE Prepares by Negotiating Powerpoint Features

In particular, she inserts a red line to indicate the cuts, and uses the resulting outlines as a guide for creating cut-out triangles. Next the TE suggests five minutes for the teachers to work on rearranging the new triangular shapes into one larger triangle. While the teachers work, the TE continues to use PowerPoint to prepare for the upcoming interactions: she draws the triangles, labels them with text numbers, and groups each triangle and its respective number together so that they can be moved concurrently (see Figure 3b). She is hopeful this will enable the teachers to more clearly communicate how to maneuver the smaller triangles into the sought-after larger one.

During the five-minute work time, three teachers volunteer to share their insights in the chat window. The TE calls on each teacher, in turn, with the hope that they could collaborate to obtain a solution. Throughout the episode, the TE and teachers use the language of geometry to communicate. For example, Layla uses the chat window to describe how to arrange triangles 1 and 2 into a square kite (see Figure 4a).

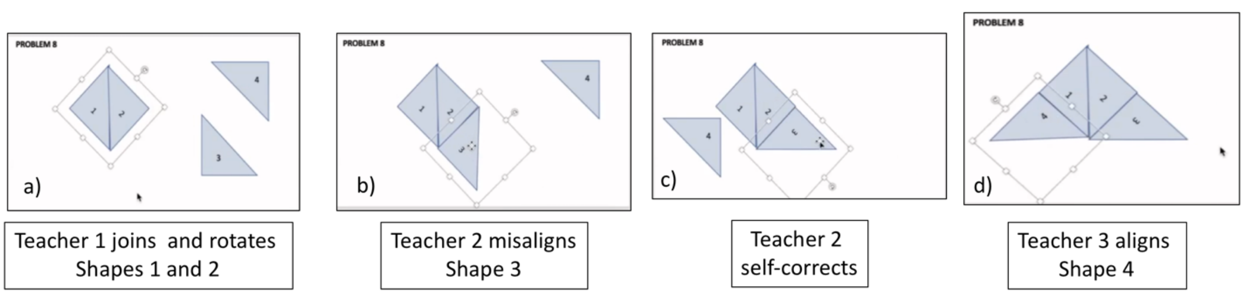


Figure 4. TE Instantiates Teachers' Instructions

In interacting with the TE, Layla gives an instruction ("keep triangles 1, 2 grouped together"), the TE instantiates the instruction, and then verifies it with her before proceeding. To ensure that the three teachers have an opportunity to contribute, the TE decides to call on another teacher, Kara, at this juncture, asking her if this coincides with her solution or if she would like to contribute something new. Speaking into the microphone, Kara verifies that she *can* build on this, and asks the TE to rotate triangle 3 clockwise 45 degrees and to align "the legs" of triangles 2 and 3. Unfortunately, the resulting arrangement doesn't match what she had in mind (see Figure 4b). "I have it on my paper and I am trying to relay. . . (Silence) I need the right angle of 3 to be up by the right angle of 2." The TE continues transforming the triangle, checking in with Kara, until the alignment matches Kara's intended solution (see Figure 4c). At this point, the TE sees an unexpected solution begin to emerge and exclaims, "I see it!" She excitedly calls on Tracy, who also uses a microphone and who successfully describes how triangle 4 can be rotated 45 degrees to complete the desired larger triangle (see Figure 4d). At the conclusion of this exploration, Layla types, "Great team work!" into the chat window.

In this episode (as in all our episodes), it is important to point out that the TE does not directly solve the problem for the teachers. Instead she is able to create an opportunity for them to investigate how *they* would solve it. Seeing this opportunity through to a successful solution entailed new negotiations for her. For example, although the TE took a chance that the three participating teachers would have the same solution, she also created an opportunity for them to articulate a different approach and *trusted* that this situation could be negotiated if it arose. Her desire to create an opportunity for interaction charges her with negotiating *layers of technology* within an online platform. That is, she works with features of PowerPoint *and* Conferences screensharing to facilitate the interactions that supported her teachers' learning. At times, she negotiates the technologies to set herself up for an exploration, ensuring that the online platform is reflecting and connecting to the teachers' actions at home (cutting, moving and aligning shapes). At other times she uses the technologies to instantiate and reflect the teachers' thinking on the shared screen. In synchronous platforms, teacher educators can be presented with a more active negotiation of *multiple* technologies, that reflect participants' problem solving, as they attempt to promote discussion and interaction.

We recognize the Conferences technology's constraints and how it precludes the teachers from working directly on the platform to present their work or to send screenshots of their solutions. However, we also saw the opportunities presented by these constraints: because the teachers were compelled to communicate solely with spoken or written words, we witnessed their struggles with the language of geometry and rotations, and observed how feedback provided to their efforts on our shared screen fostered learning. In effect, the teachers in this episode were able to interact with the content through the TE using the Conferences technology, and the shared technology workspace provided necessary feedback for them to move the solution forward. This illustrated additional kinds of interactions that can be supported in online teaching and learning (see Swan, 2002). This observation is especially significant when the students in an online platform are *teachers* who must not only model the careful use of mathematical language for their own students, but listen carefully to their students' mathematical reasoning and use it to support their interactions and learning (Pimm, 1987).

The Dollar Bill Episode

In the Dollar Bill episode, we present a richer and more complex scenario of the TE employing multiple technologies and physical spaces to engage mathematically with the teachers. In the episode, the TE tasks the teachers with estimating the length and width of a dollar bill using multiple units of measurement. This activity intended to build on prior estimation activities and deepen the teachers' understandings of relationships that preserve and vary under measurement. The teachers had been instructed to have a dollar bill and a ruler with centimeters and inches handy. The TE had posted a Powerpoint, before their meeting, whose images included a dollar bill and ruler, as well as a dollar bill with hair pins and paper clips all drawn to scale (see Figure 5).

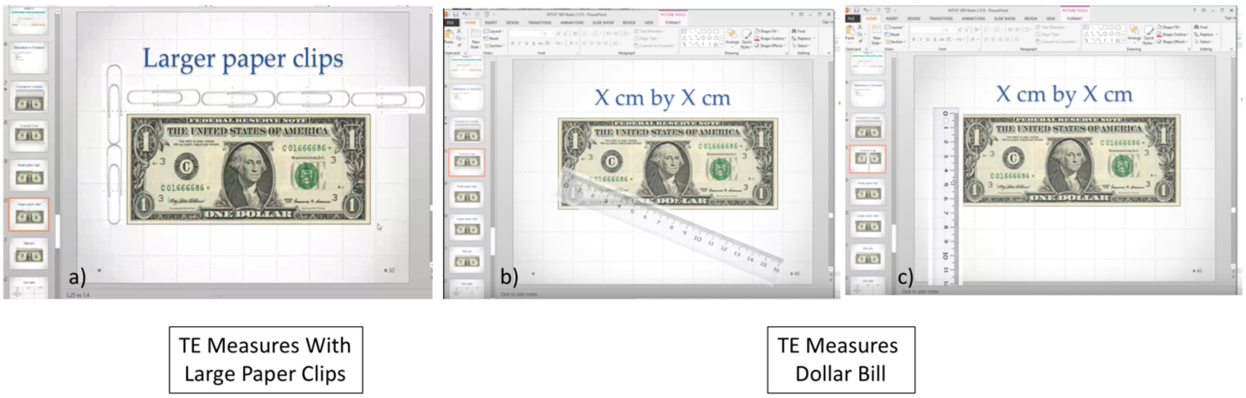
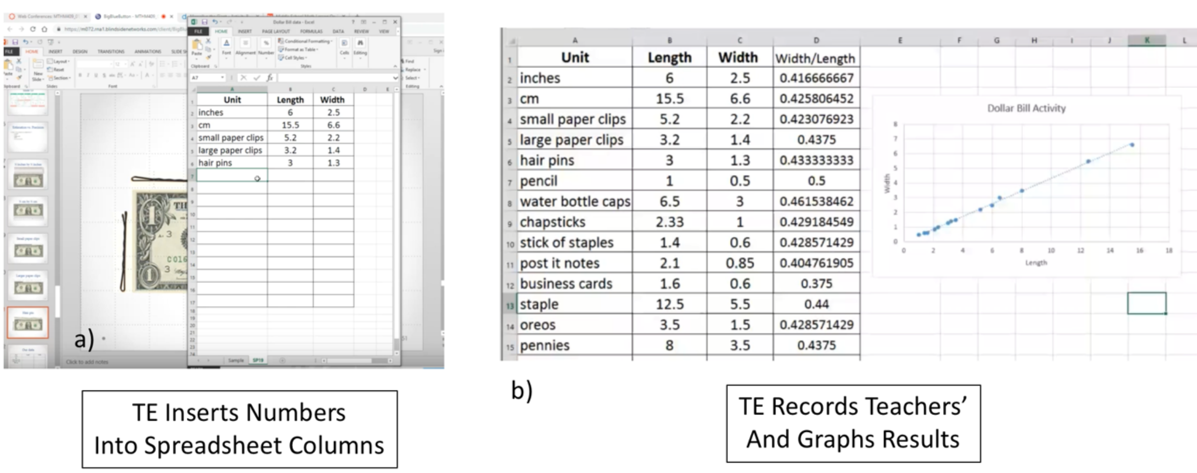


Figure 5. The Dollar Bill Episode

The TE displays a dollar bill on an un-played PowerPoint slide and introduces various units to measure the bill's dimensions (see Figure 5). She asks the teachers to measure their dollar at home with a given unit (like centimeters), elicits their guesses, interpolates a response, and enters the response into the Length and Width columns of a spreadsheet (see Figure 6a).



Unit	Length	Width	Width/Length
inches	6	2.5	0.416666667
cm	15.5	6.6	0.425806452
small paper clips	5.2	2.2	0.423076923
large paper clips	3.2	1.4	0.4375
hair pins	3	1.3	0.433333333
pencil	1	0.5	0.5
water bottle caps	6.5	3	0.461538462
chapsticks	2.33	1	0.429184549
stick of staples	1.4	0.6	0.428571429
post it notes	2.1	0.85	0.404761905
business cards	1.6	0.6	0.375
staple	12.5	5.5	0.44
oreos	3.5	1.5	0.428571429
pennies	8	3.5	0.4375

Figure 6. Negotiating Spreadsheet and Powerpoint in Dollar Bill Episode

She then “measures” the dollar with the given unit on a PowerPoint slide, that is, she aligns the “unit object” alongside the bill’s length and width, demonstrating how it can be used to approximate the dimensions (see Figure 5b and 5c). It is important to re-emphasize that the TE had designed each slide’s images to accurately represent the real-life measurements being taken at home. A back and forth interaction ensues in which teachers guess the bill’s length and width using assigned units like inches, centimeters, paper clips and hair pins. With each unit, the TE toggles back and forth between the display containing the PowerPoint slide and the spreadsheet (see Figure 6a) as well as the chat window displaying the teachers’ input.

The TE next opens up the measurement exercise by asking the teachers to look around where they “are sitting right now” and “find some non-standard unit” to measure the length and width of a dollar bill. The teachers volunteer estimates using water bottle caps and staples among other objects. This section of interactions did not contain verification measurements, which sometimes led to disagreements, struggles or negotiations among the teachers or between the TE and teachers! For example, Leann approximates the bill’s length to be 2 post-it notes, but does not provide the width. The TE elicits the width from her, and fills in the unit name on the spreadsheet, but not the dimensions. As the TE awaits Leann’s complete response, other unit suggestions appear which scroll the chat window until Leann’s length suggestion disappears. In the meantime, Brandon volunteers 2.1 by .85 post-it notes which is entered into the spreadsheet (see Figure 6b). The TE recalls Leann’s use of a post-it, and asks her if she agrees with Brandon’s width estimate. Leann types “5/6 of a post-it note” into the chat window and the TE remarks that 5/6 is close to .85 but also asks her to “just remind us the length that you found for the post-it note.” In response, Leann volunteers “76 mm” or “7.6 cm.” Now the TE recognizes a discrepancy with Brandon’s dollar bill length. When the TE asks Leann again to confirm or refute Brandon's measurements, Brandon then volunteers, "it's like a hair longer than 2 post-it notes" and continues typing "but we could say 2, that's fine." As he types, the TE wonders aloud whether Brandon and Leann are using the same size post-it notes and she opts to continue eliciting other estimates. But Brandon disagrees with her and types, "nah we have the same standard note."

This episode concludes with the TE asking the teachers what they could do with the spreadsheet table. Teachers suggest finding areas, graphing length and width, dividing length and width, and Brandon types, "width is all about 0.41 of the length for all units … or length is 2.4 times the width … either way." The TE uptakes the division suggestion and enters a formula that divides the width by the length entry into the spreadsheet cell (see Figure 6b). One by one, she drags the formula into the subsequent column cells, confirming Brandon's observation. The results surprise and intrigue the students who type into the chat, "I thought they would all [ratios] be the same but you can see clearly for some of them it would not be equal," "no matter what units we are using, they all should be proportional," and "it's cool no matter what unit you are using so similar even though we are estimating … ha." The TE was equally delighted to hear students' responses to her question: "What do you think is going to happen when I try to graph the length and the width? What will this picture look like?" More guesses are volunteered ("somewhat close to a straight line" and "linear because they are proportional") and the TE's graph is used to confirm their guesses (see Figure 6b).

This energetic episode illustrates possibilities for more complex synchronous interactions in investigating mathematics problems. The TE's efforts are reflected, in part, by her preparation of slide images whose dimensions indicate the actual measurements of the objects being studied. This, in turn, transforms the slide into a setting whose activities mirror some of the measurements the teachers were taking at home, and reflects the teachers' thinking about how to measure the dollar bill with a variety of objects. As in the Two Squares-One Triangle episode, both the home and technology platform environments and activities converge to move the problem solving forward, although the PowerPoint slide preparation in the current episode is more nuanced and complex. Within the Conferences platform, the TE negotiates PowerPoint, a spreadsheet and the Conferences chat window to verify guesses, illustrate measurement approaches, organize and record responses, and communicate with the teachers.

Opening up the unit selection to the teachers' surroundings communicates the TE's trust in the teachers' learning, and in their collaborative ability to negotiate their contributions. This becomes especially evident in the interaction involving Leann, Brandon and herself: Leann and Brandon's negotiation of the post-it dimensions for the dollar bill appear, at first glance, to generate a disagreement over the bill's length (Leann suggests 76 mm, which is less than the length of 2 post-its that Brandon suggests). However, reflecting on the episode discloses that Leann's 76 mm may have been referencing the length of a *post-it* and not the dollar bill, particularly given its response to the TE's elicitation for the "length" of "the post-it note." This casts Brandon's participation in this interaction in a new light: his offer to sacrifice the ".1" in his 2.1 estimate can be interpreted, in retrospect, as a gesture for supporting Leann's earlier estimate of 2. The TE recalls balancing "respect for and promotion of student thinking" within this online interaction "with progress toward mathematically sound conceptual understanding" (Trenholm et al., 2016, p. 149).

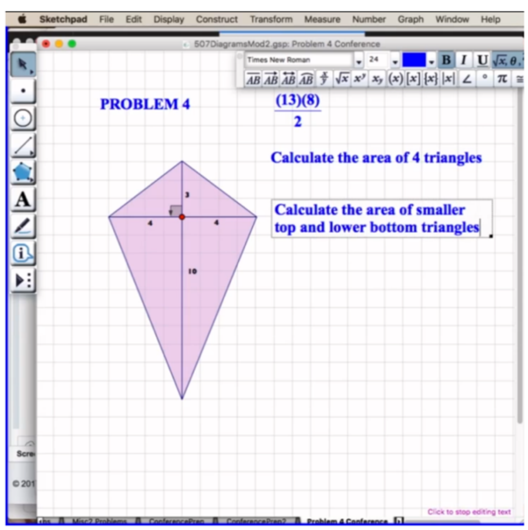
Through the current study's reflections, the TE realized she had confused Leann's 76 mm for the dollar bill length and she considered how the chat window's scrolling may have contributed to this oversight. These discoveries begin to convey the value in a TE's using synchronous recordings as a tool for reflecting on the teaching and learning of mathematics (Trenholm et al., 2016). We conclude by noting that the TE had made this interaction a safe place for Brandon to agreeably disagree with the TE's conjecture about Brandon and Leann using different-size post-its.

An Evolving Lesson: the Teachers' Influence on Lesson Development

In considering the possibilities for demonstrating the value in synchronous interactions, we also examined situations in which the teachers' observations drove, and sometimes altered, the TEs' planned lesson. In these episodes, the reader will note how teacher-initiated insights helped to refashion a lesson's intended goals.

The Kite Episode

In the geometry class, the TE and teachers worked together on a homework problem asking for six different approaches to finding the area inside a kite (see Figure 7).



The Kite Problem and
Record Keeping of Strategies

Figure 7. The Kite Problem

A few teachers, including Emily, had indicated they had obtained only one or two approaches. The TE is seen making use of three technologies in this episode: Geometer's Sketchpad (GSP), PowerPoint, and the chat window, switching between the three as needed.

In the first 21 minutes of exploration, the teachers suggested and generated three approaches the TE expected, all of which were justified and recorded on the shared GSP workspace (see Figure 7). That is, the TE and teachers justified the kite's area formula, which uses half the product of its diagonals, and investigated how two of the kite's triangle decompositions also could be used to find the kite's area. During one of the triangle decompositions explored in GSP, Megan uses the chat window to pose the following question, "Can we transform the two smaller triangles into a square and then the two larger triangles into a rectangle and then add the area of the square and rectangle together?" As the TE exclaims, "Oh wow, Megan's, hold that thought," Brandon types, "That's what I did Megan (4x3) + (4x10) = 52." Eager to be guided by Megan's words, and anticipating the chat's scrolling feature, the TE takes a screenshot of Megan's chat window entry and inserts it into a partly-prepared PowerPoint slide (see Figure 8a).

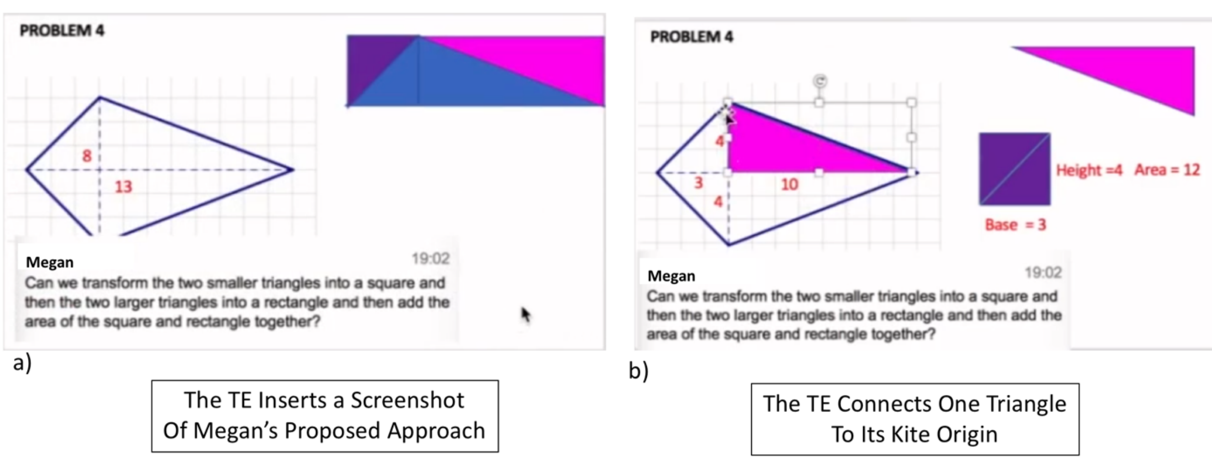


Figure 8. Megan's Approach for Kite Area

Her decision to change workspaces (from GSP to PowerPoint) is mostly guided by her uncertainty over how to insert screenshots into the GSP workspace.

In order to investigate Megan's idea, the TE makes use of a rectangle she had created for the earlier area investigation using the kite's diagonals (in Figure 8a, see the upper right corner). She had constructed this rectangle from the kite's four component triangles and uses these triangles to begin constructing Megan's re-arrangement. The TE recalls feeling comfortable in her understanding of Megan's question, in part because so much time had been spent earlier investigating the kite's decomposition into possible triangles. As the TE re-arranges, Tracy questions part of Megan's entry, typing, "Technically it would be two rectangles, right?" and noting, "If we used the dimensions from Problem 4, it would be a 3x4." Tracy appears to be clarifying that neither shape will be a square and the TE responds, "I think you are right." The TE recalls thinking it would be important for the teachers to justify the resulting quadrilaterals' dimensions as part of their investigation and she is careful to elicit these justifications in her interactions (eg., "I need somebody to tell me what the dimensions are of the resulting shape" and "What's a good way to explain this to the rest of the class?"). She uses PowerPoint's color feature to distinguish the newly formed rectangles, and as she re-arranges triangles, she is careful to connect these two shapes to the original kite (see Figure 8b). At the investigation's conclusion, the TE praises Megan's observation, which adds a fourth approach to the running tally. She restates the original goal of finding six approaches for the kite's area, and modifies it to, "We need two more, actually we need three more because I already had six and I hadn't thought of Megan's!"

For the TE, the task of investigating the kite's area using six different approaches created an open-ended situation for multiple students who had come up with only one or two solutions.

By the conclusion of the investigation, she felt satisfied that the work done had encouraged the teachers to look at the parts of a kite dynamically and to see how geometry provided a lens into how formulas could be understood from decomposing and re-arranging geometric shapes. At the end of the episode, Emily noted, “I was just so focused on using the few ways that I know that I couldn’t outside (sic) of the box to find other ways.” For the TE, creating an opportunity for “outside the box” thinking through synchronous interactions was rewarding.

These interactions also introduce an opportunity to discuss the TE’s public admission that she had not considered this teacher-initiated perspective on the kite’s area. As in the Two Squares-One Triangle episode, this same TE is faced with an unexpected solution method to a posed task. Working with “the unexpected with a genuine interest in learning its character, its origins, its story and its implications” (Confrey, 1990, p. 108) is one of the classroom interaction phenomena that mathematics educators celebrate in face-to-face settings (Lampert, 1990). In Two Squares-One Triangle, the collaboration that went into developing the solution was celebrated by the TE and at least one of the participants; in the current episode, the teacher’s contribution helped to shape the lesson development by creating an opportunity to investigate the approach and to modify the problem statement as a result of the investigation. Both episodes provide descriptive evidence of the valuable role the unexpected can continue to play in synchronous online settings.

And yet the complexity of interactions admits struggles and oversights as well. Our reflections on this lesson’s recordings revealed an error in the TE’s pre-prepared PowerPoint slide: the dimensions on the kite’s smaller right triangles were incorrectly drawn as 3, 3 instead of 3, 4. This oversight likely contributed to Megan’s observation that part of her decomposition would result in a square. We recognize again the value in using synchronous interaction recordings to reflect on teaching and learning and to assist a TE's evolution as an online instructor. We note additional growth in the TE's role in synchronous interactions through the insertion, into the workspace, of a screenshot of a teacher's chat window observation. This TE had learned that taking these screenshots helped to mitigate the chat window's scrolling feature, but more powerfully, it also helped to keep that teacher's voice at the forefront of the investigation.

A Culminating Episode: The Two Cylinders

We conclude our data presentation with an episode we title A Culminating Episode because it captures so many of the central interactional elements we investigated above. For the reader wondering about the complexity of instantiating these elements in one investigation, we encourage you to read the Cylinders episode and consider how the TE:

• creates opportunities to work with questioning, guessing, disagreeing, struggling, negotiating, explaining, and justifying,

• works within a problem situation that was open-ended for her teachers,

• enables teachers to work together to resolve issues,

• negotiates multiple technologies (PowerPoint, Geogebra, Equation Editor and the chat window),

• connects the home and technology workspace environment (through the use of isometric dot paper and Geogebra),

• builds on a teacher's unexpected insight to move the lesson forward (when a 2D interpretation is offered to explain the class's misconception), and

• illustrates a new useful online strategy (when the TE interacts by typing into the chat window).

The Two Cylinders Episode

In the Two Cylinders episode, the TE aimed to implement student-driven instruction to achieve pre-planned learning objectives. Enacted as part of a unit that required reasoning with and about surface area and volume formulas for prisms and cylinders, the TE intended to “move beyond calculations” and consider “hidden relationships” among length, area, and volume concepts. To this end, she asked the class to assess the validity of a claim about two cylinders. Specifically, the teachers needed to decide if the can for Yummy Soda contained twice as much soda as Good Soda (see Figure 9a and see Parks, Musser, Trimpe, Maurer, & Maurer, 2007).

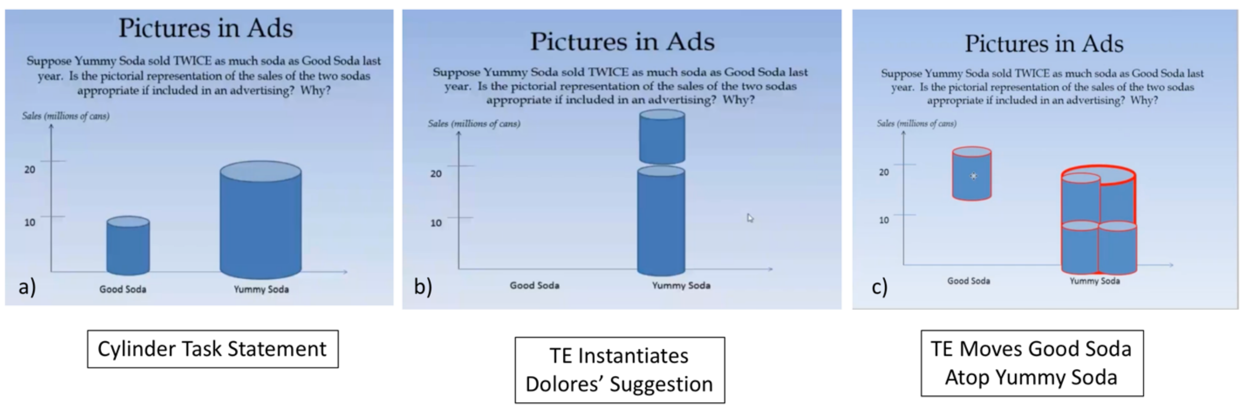


Figure 9. The Two Cylinders Episode

The TE presented this scenario with a PowerPoint slide in Normal View mode, where the cylinders were inserted separately using PowerPoint shapes feature. This enabled the TE to later adjust their size and location, based on the teachers’ directions and suggestions.

With the question, “Is this picture a good way to represent this situation?” the TE aimed to elicit evidence of teacher thinking about the dimensions of a cylinder. In response, teachers began to point out height and width differences among the cylinders using the chat window (eg., “it looks a lot more than twice the amount because the height doubled but the radius is also larger” and “I would say no because it looks like the bigger cylinder is 4 x the size of the smaller"). One student, Elizabeth, focused on both height and width when she said, "I don't think this picture is good because it doubles in both width and height. So it's 4 times the amount." Several teachers agreed (eg., "4 Good Soda could fit into the Yummy Soda"). No other possibilities were proposed, and some teachers refrained from answering.

Throughout this process, the TE reminded herself to allot appropriate wait-time for the teachers to study the picture. She understood the emerging misconception about a cylinder's "missing" dimension and recognized the length versus area connections being applied by the teachers. She moved the image of Good Soda on top and then next to Yummy Soda to show side-by-side views of the respective widths and heights. These efforts appeared to have limited effect at first, until Dolores typed that "yummy soda would need to be thinner in order to be a better representation." The TE responds, "Dolores, if you tell me how to change the shape of Yummy Soda, I will." Here, Dolores turns on her microphone and asks the TE to "squeeze in Yummy Soda until it was the same size as Good Soda." The TE completes this in PowerPoint (see Figure 9b), and Dolores continues speaking, "That shows twice as much (...) now you would have two cans for the one." With this exchange, the class comes to agree that the volume of a cylinder can double when only the height doubles, but that, "the [larger] size of the base was making it 4 times as big" (Dolores). In order to consolidate all the teachers' observations, the TE moves the investigation forward by asking, "If the height and the radius of a cylinder double, does that quadruple the volume?" When the teachers do not respond, the TE wonders aloud about the meaning of their silence ("I'm wondering what you are thinking"). She recalls wondering whether they were reconsidering their claims, rethinking the impact of a radius (versus a diameter or width) on volume, or if they felt they had already answered the question.

As the TE moves away from the unresolved cylinder task, one teacher types in her confusion, “I’m still not sure to be honest.” The TE recognizes this remark, and says, “Let’s look into this a little bit more.” She decides to introduce a new investigation to help the teachers either develop their arguments or reconcile uncertainties in their thinking. To this end, she uses Geogebra’s isometric grid feature (geogebra.com) to sketch a model of a 2 cm by 4 cm by 6 cm prism. The teachers were encouraged to do the same at home using Geogebra or isometric dot paper (which they had been instructed to print out for the lesson). For the prism investigation, the TE joins the teachers in the chat window, typing, “How should I double the volume?” Teachers suggest, “double the height, width, or length.” She then considers the effect of each dimension change in turn (first the width, then the width and length, and finally all three dimensions). For each case, she elicits a guess, constructs the effect on Geogebra, and discusses with the teachers their predictions (see Figure 10).

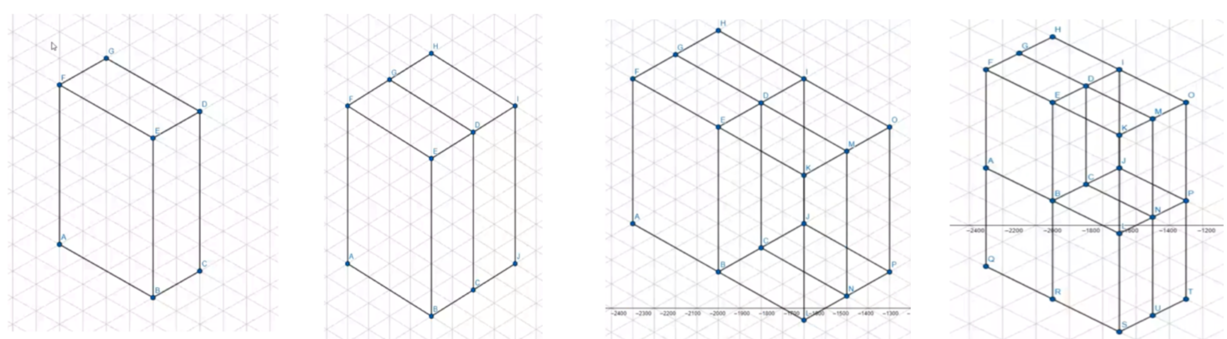


Figure 10. Changing Prism Dimensions in Geogebra (not drawn to scale)

For example, asking about the effect of doubling the width and length, teachers predict that volume “would double twice” or “would be 4 times bigger.”

In response to the question, “What if we double all 3 dimensions: width, height and length?”, the teachers agreed with one another that the volume of the larger prism would be 8 times that of the original prism. Two of the participating teachers predict this before the Geogebra construction is completed, while others elaborate their thinking later ("I think it's an exponential growth" and "8x"). Throughout the prism exercise, the TE recalls struggling with toggling between the chat window and Geogebra because the Geogebra images would disappear while she was in chat mode (this is because Geogebra was opened in the same browser as Conferences). Nevertheless, she continues to switch between the platforms to maintain her interactions with the teachers, reading aloud and discussing comments shared by the teachers in the chat window.

As the TE was closing this activity in Geogebra, Tracy shared the following insight in the chat window: "Okay, so I'm thinking about the cylinder from before - I think it's 8x the volume." Upon her return to the chat window, the TE begins to read Tracy's text entries, as they are being typed in one line at a time:

> Okay, so I'm thinking about the cylinder from before – I think it's 8x the volume
> Because when we are still seeing a circular cylinder
> not a cylinder that's just wider, it's still circular
> which means the radius that extends horizontally and then depth-wise,
> the radius also doubled.

Taken aback at the clarity and significance of Tracy's thinking, the TE acknowledges it along with the thinking shared by others saying, "Some of you are beginning to think of the cylinder and I'm so glad that you are thinking of a cylinder right now" and "The height has definitely doubled, and what else doubled?" In response, a student types, "the width, the radius." The TE builds on all these insights, switches to the Equation Editor, and constructs with the teachers an algebraic representation and justification of their geometric thinking (Volume of

Good Soda = $\pi r^2 h$ and after doubling the radius and height, Volume of Yummy Soda = $8\pi r^2 h$). These investigations lead teachers to revise their initial conjectures (eg., "So if we double the height and the radius it is actually 8 times as big not 4") and feel like they bring the investigation to a close. Nevertheless, the TE challenges the teachers one last time by asking, "Why do you think so many of us thought that Yummy Soda was four times bigger than Good Soda?" Tracy types, "… because we're not used to think (sic) about how the circle doubles in size … it's easier to imagine it when you think about the length and width of a rectangle but the 'length' and 'width' of the circle is essentially the radius." Elizabeth proposes, "We were thinking in 2D figures, not in 3D." The former insight captures the impact of changing a circle's radius as effecting a length *and* width change; the latter captures the misconception effected from reasoning from a two-dimensional representation. Five teachers support Elizabeth's observation through their chat contributions and frame the TE's upcoming response when she substantiates their perspective, extemporaneously, by moving four cans of Good Soda "atop" Yummy Soda (see Figure 9c).

For the TE, the challenges of participating in these online explorations were far offset by her witnessing the development in her teachers' thinking, the insightfulness of their observations, and their willingness to take on (what was for them) an open-ended problem situation. By enabling the teachers to make informed and public conjectures, to discuss and test these conjectures, to revise them in response to investigations, and to justify them, the TE lays a "major pathway to discovery" (NCTM, 2000, p. 57). Despite the difficulties experienced negotiating the Geogebra and Conferences workspaces, she persevered through "pedagogical discomfort" faced as a result of her "managing a new learning environment" in an online setting (Frykholm, 2004, p. 133). As a result of her efforts, the teachers were able to engage mathematically with "productive wrong ideas" (Duckworth, 2006, p. 70) and advance discoveries about relationships between a cylinder's dimensions and its corresponding volume.

DISCUSSION

As teacher educators, we begin by noting the features of our online practice that stayed constant in transitioning from our face-to-face experiences. Working to uphold constructivist principles and support real-time interactions remained an ongoing endeavor in this new setting. For us, choosing worthwhile problems and mathematical tasks also remained as crucial in online settings as they were in face-to-face ones (see NCTM, 2000). Problems that were open-ended or that invited multiple approaches or that encouraged teachers to examine their thinking in relation to how mathematics investigations developed were essential components in engaging mathematically with our teachers. We note how *bridging* our beliefs and tasks was critical to adapting posed problems to the ongoing demands presented by the TEs' and teachers' contributions. Again, as in face to face settings, taking on these challenges involved risk-taking and admitting that one's assumptions and insights are open to revision and discussion as we engaged mathematically (see Lampert, 1990).

In online settings, risk-taking seems especially significant when discussing the subject of technology. In contrast to the finding that a synchronous virtual classroom is "easy to use," (Martin and Parker, 2014, p. 201), we experienced novel and unprecedented challenges in using technologies to bring geometry and measurement to our teachers. We believe this observed discrepancy may be related, in part, to our desire to encourage interactions while engaging mathematically with teachers as opposed to "giving" them answers or "telling" them how to solve problems. Technologies with which we had extensive experience (like PowerPoint and GSP) were modified dramatically for use in our online classes. For example, we never used the

PowerPoint application as a "Slide Show." Instead, we used it as a working tool in "Normal View" mode and treated each slide as a "work in progress." This meant that our slides became spaces where we could add and format text to reflect observations, questions and insights; add, group, and manipulate shapes; reflect, translate or rotate shapes; add color; insert screenshots; and write mathematically representative equations. We characterized this experience as one of *repurposing familiar technologies*, that is, we adapted familiar technologies for use in an online setting so that interactional elements could be supported as we engaged mathematically with our teachers. This experience enhanced our teacher knowledge base as we learned the new capabilities of these familiar technologies for the purpose of using them in online teaching and propelled us to re-think how our teaching would change as a result of repurposing these tools (Mishra and Koehler, 2006).

In addition to repurposing familiar technologies, we also learned new technologies. In particular, we learned how to work with the Conferences synchronous platform within our university's Canvas Learning Management System. In another contrast to a finding that synchronous virtual classrooms require "minimal training" (Martin and Parker, 2014, p. 201), we engaged in months-long training with Conferences. We practiced with it and discussed it amongst ourselves and with our university's information technology experts so that we could optimize its use in engaging mathematically with teachers. When we integrated our repurposed technologies and its newly discovered capabilities into this new platform, we found ourselves continuously negotiating multiple layers of technology in our teaching. We worked through constraints as they arose, sometimes accepting their limitations (like shouldering Conferences screensharing restriction of moderator privileges to the "instructor"). At other times, we perceived opportunities within these constraints which could lead to learning (like utilizing this restriction so that other layers of technology could provide feedback and clarification in response to participating teachers' communication efforts). And at still other times, we created new teaching strategies to accommodate constraints (like inserting screenshots of teacher text entries into our working spaces to mitigate the chat window's scrolling feature). All these negotiations illustrate how technologies can change how we teach the subject matter (see Mishra and Koehler, 2006), and ultimately, how our teachers learn.

One of our two most surprising findings in embarking on this experiment concerned our discovery about *environments* in online teaching. As we taught and interacted with teachers within this synchronous platform, the definition of *environment* came into question. Turkle (1997) suggests that as "humans are mixing increasingly with technology and with each other through technology, what distinguishes the specifically human from the specifically technological becomes more complex" (Rosa and Lerman, 2011, p. 82). We experienced this complexity and viewed it as enriching our opportunities to engage mathematically with our teachers. For our teachers and ourselves, we came to understand that we would be occupying, and working with, both a *synchronous platform* and the *space in which we were using that platform*. The synchronous platform (which was new to us) was a technological workspace—a place where insights and modifications could be recorded and used to reflect and contribute to ongoing interactions. However, we also considered the space in which we were using this platform to be a resource in teaching and learning (eg., our homes, our offices, a classroom). That is, within our respective spaces, we could work with concrete, pre-arranged manipulatives (eg., cut-outs of shapes) and even draw immediate materials that could be connected to the ongoing lesson (eg., objects to measure). It was critical that we uphold the connections between these workspaces (by preparing careful slides, or sending teachers worksheets with shapes or with dot paper) in order to negotiate the ongoing interactions. Because our students and we were occupying and working with *both* the technology and our spaces in novel ways, we viewed our negotiations in these interactions as worthwhile in advancing the literature on online teaching.

Our second surprising discovery concerned the use of our recordings in our own professional development. By listening to and watching the recordings, independently and together, we were able to articulate and refine teaching strategies for later online experiences. We developed more enhanced understandings of some of the struggles our teachers had with lesson content and this often led to revisions in lesson materials or in our understandings of teachers' thinking. The ease with which an online lesson can be recorded (all you have to do is click a button) suggests vast opportunities for reflecting on and using these videos in teacher education practice and research. From a practical stance, we underscore how critical collaboration was in our experiences. Duckworth (2006) notes, "most teachers need the support of at least some nearby co-workers who are trying to do the same thing, and with whom they can share notes." (p. 9) and NCTM (2014) emphasizes that, "watching and critiquing instruction with colleagues by using video clips can be one of the most effective ways to promote reflection, growth, and learning" (p. 105). We strongly relate to, and encourage, working together if teacher educators decide to embark on this kind of transition.

CONCLUSION

This self-study describes the negotiations and strategies of two teacher educators attempting to engage mathematically with teachers through purposeful and mindful interactions. By conveying "images of the possible" (Shulman, 2004, p. 147), we hope we have demonstrated to the mathematics teacher education community that synchronous online interactions *can* be used to support experiences for teacher education practitioners to engage mathematically in this new platform. By substantiating our lesson episodes with our insights and our training experiences, we hope we have brought an enhanced understanding to at least part of the work involved in our online endeavor.

The value of our work to the mathematics teacher education community can be appreciated from many perspectives. Standards for preparing mathematics teachers encourage experiences in which teachers can experiment with the role of technology in their teaching, and urge teacher educators to craft such experiences (Association of Mathematics Teacher Educators, 2017). Learning and teaching online provides one novel venue for just such an experiment. Through our efforts to teach mathematics synchronously, teacher educators can serve as possible role models who dare to attempt new teaching and learning formats while striving to create meaningful opportunities for engaging mathematically. The successes of these endeavors and the demonstrations of weathering the challenges and constraints of these technological tools and environments are *shared* between teacher educators and teachers in a way that other technology experiences are not. We also can imagine lesson recordings from online lessons being used by participants to reflect on teacher education experiences in novel ways. For developers of learning management platforms, we anticipate our research informing features that will work to facilitate (versus limit) how online interactions can promote engaging mathematically. For the mathematics teacher education research community, our work can motivate research questions such as: What multiple purposes do chat communications serve in synchronous interactions as participants engage mathematically? How do teacher educators lesson plan for engaging mathematically in their synchronous online teaching? What is the student perspective on engaging mathematically in these synchronous interactions?

As rapidly evolving technologies continue to expand online teaching into more and more educational venues (Trenholm et al., 2016), it becomes more important than ever for the teacher education community to participate in this work and *define* (not follow) the directions that ensure interactions remain a key component in the experiences of mathematics teacher education. The vast number of opportunities for learning about teaching, for professional development and for online developers afforded by modern technologies makes online instruction a particularly promising platform to the field of mathematics teacher education.

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KEY TERMS AND DEFINITIONS

Asynchronous Online Interactions: Interactions mediated via a technology forum which provide an opportunity for students and teachers to interact with each other possibly at different times

Constructivism: A theory of learning that recognizes that knowledge is not passively received, but actively constructed, by the learner

Engage Mathematically: A teacher and her students engage mathematically when their investigations into a mathematics problem are not prescriptive or the outcomes are unknown, and the teacher supports students in exploring the problem rather than dismissing, overlooking or immediately resolving it

Environment: Physical or virtual workspaces that can be used and connected during online teaching

Layers of Technology: The presence of multiple technologies like screensharing, Powerpoint, and Geogebra that can interact during the course of online teaching

Repurposing Familiar Technologies: Adaptation of familiar technologies (eg., PowerPoint, Geogebra) for use in an online setting so that interactional elements can be supported as students engage mathematically with the teacher

Synchronous Online Interactions: Interactions mediated via a technology forum in which participants are present at the same time, just not necessarily in the same place.