5D IIR and All–Pole Lattice Digital Filters

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***Abstract*—The absolute minimum number of delay elements characterizes the implementation of this newly proposed five dimensional (5D) circuits, having infinite impulse response (IIR), as well as all–pole digital filters. For both structures, filters having the lattice property are considered. The 5D circuit realizations use, for their implementation, a minimum number of delay components. In addition, the dimension of the state–space vector, of this particular implementation of the 5D model, is minimal. The minimization of circuit implementations and their realizations in state–space is demonstrated with two illustrative examples.**

***Index Terms*—5D filter, 5D system, IIR lattice filter, all–pole filter, state–space, realization.**

I. Introduction

In recent decades, for long periods of time, the field of multidimensional systems and filters, has been at the center of electrical engineering with an emphasis in digital signal processing [1]–[6]. Extensive related research efforts, in this area, are the study of five dimensional (5D) systems and filters. This particular area has already contributed significantly to the processing of field light video using IIR filters, IIR depth velocity filters, printed circuit boards testing, adaptive depth velocity filters, and FPGA circuits [7]–[9].

In this paper, generalized circuits and minimal state–space realizations for 5D IIR, and all–pole lattice digital filters are introduced. In addition to hardware limitations, the necessity for minimal implementation results from the absence of the fundamental theorem of algebra for polynomials with more than one dimension, which might result in theoretical or computational complications when non–minimal implementations are utilized [2], [3].

Only for filters and systems up to four dimensions (4D), minimal circuits and state–space realizations have been reported [10], [11]. The results of this paper are the first attempt to address the problem of minimal realization in 5D. These digital filters in 5D are more complex, and require twice as many coefficients as 4D filters.

Notwithstanding the difficulties in deriving the 5D transfer functions, the end result is very elegant and follow the well–known all–pass patterns, and properties of the conventional one dimensional (1D) structures. What is interesting here is that the dimensions increase from 4D to 5D, the diamond type resulting transfer function structures appear to be more symmetric or nearly symmetric.

II. Problem Statement

The following 5D (five dimensional) state equations are comprised of a cyclic 5D structured state–space model [12], [13].

$$\dot{\mathbf{x}}(i_1,\cdots,i_5) = \mathbf{A}\mathbf{x}(i_1,\cdots,i_5)+\mathbf{b}u(i_1,\cdots,i_5) \quad (1)$$

$$y(i_1,\cdots,i_5) = \mathbf{c}'\mathbf{x}(i_1,\cdots,i_5)+du(i_1,\cdots,i_5) \quad (2)$$

where,

$$\mathbf{x}(i_1,\cdots,i_5) = \begin{bmatrix} x_1^{d_1}(i_1,\cdots,i_5)\\ x_1^{d_2}(i_1,\cdots,i_5)\\ x_1^{d_3}(i_1,\cdots,i_5)\\ x_1^{d_4}(i_1,\cdots,i_5)\\ x_1^{d_5}(i_1,\cdots,i_5)\\ \cdots\cdots\cdots\cdots\\ x_{5n}^{d_1}(i_1,\cdots,i_5)\\ x_{5n}^{d_2}(i_1,\cdots,i_5)\\ x_{5n}^{d_3}(i_1,\cdots,i_5)\\ x_{5n}^{d_4}(i_1,\cdots,i_5)\\ x_{5n}^{d_5}(i_1,\cdots,i_5)\end{bmatrix}, \quad (3)$$

$$\dot{\mathbf{x}}(i_1,\cdots,i_5) = \begin{bmatrix} x_1^{d_1}(i_1+1,i_2,\cdots,i_5)\\ x_1^{d_2}(i_1,i_2+1,\cdots,i_5)\\ x_1^{d_3}(i_1,i_2,i_3+1,i_4,i_5)\\ x_1^{d_4}(i_1,\cdots,i_3,i_4+1,i_5)\\ x_1^{d_5}(i_1,\cdots,i_4,i_5+1)\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ x_{5n}^{d_1}(i_1+1,i_2,\cdots,i_5)\\ x_{5n}^{d_2}(i_1,i_2+1,\cdots,i_5)\\ x_{5n}^{d_3}(i_1,i_2,i_3+1,i_4,i_5)\\ x_{5n}^{d_4}(i_1,\cdots,i_3,i_4+1,i_5)\\ x_{5n}^{d_5}(i_1,\cdots,i_4,i_5+1)\end{bmatrix}. \quad (4)$$

The matrices $\mathbf{A},\mathbf{b},\mathbf{c}'$ of the above 5D state–space model (1), (2), that is used to represent our 5D digital filter, have the dimensions: $(5n\times 5n)$, $(5n\times 1)$, $(1\times 5n)$, and d is a scalar.

The direct application of the 5D z–transform on (1), (2), returns the following transfer function:

$$H(z_1,z_2,z_3,z_4,z_5)=\mathbf{c}'[\mathcal{Z}-\mathbf{A}]^{-1}\mathbf{b}, \quad (5)$$

where, $\mathcal{Z}=z_1\mathbf{I}_n \oplus z_2\mathbf{I}_n \oplus z_3\mathbf{I}_n \oplus z_4\mathbf{I}_n \oplus z_5\mathbf{I}_n$, with $\oplus$ denoting the direct sum.

The following sections present 5D lattice circuit structures, with minimum delay elements, and cyclic state–space realizations, with minimum state vector.

III. 5D FIR Lattice discrete filters

Based on the theory of conventional lattice filter [3], [4], the circuit implementation of a 5D lattice FIR digital filter is shown in Figs. 1 and 2.

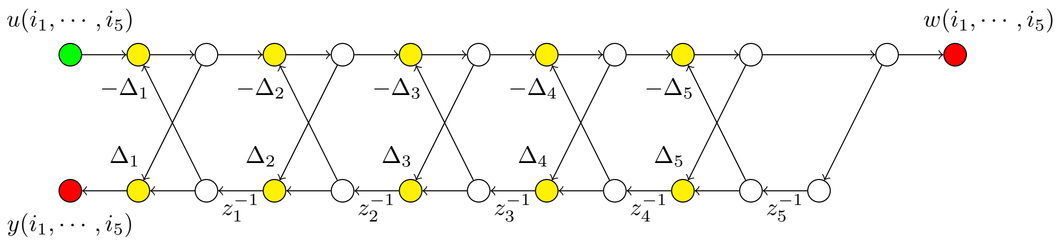

Fig. 1: $[S_i]$: 5D first–order IIR digital lattice filter.

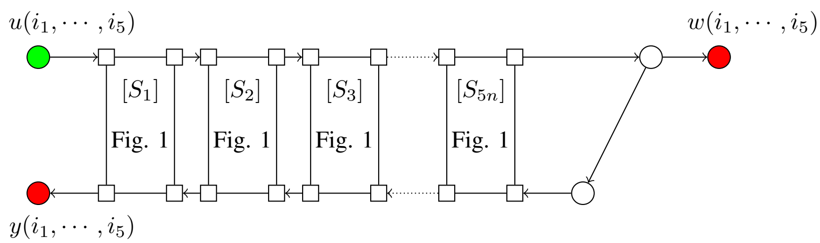

Fig. 2: 5D Generalized IIR lattice digital filter structure. Each square is a copy of the single section of Fig. 1.

The proposed circuit with $5n$ delay elements, $10n$ multipliers and adders is illustrated in Fig. 1.

Next, the 5D cyclic structured state–space model $(\mathbf{A},\mathbf{b},\mathbf{c}',d)$ will be derived.

To obtain the state–space equations, for the 5D model, ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}'$, d), the following procedure is used: "Label the outputs of each delay element (in Figs. 1 and 2) to indicate the model's states. Write a state equation, by inspection, for each delay element. Rearrange the equations so that each block contains all of the state variables that are accessible. Extrapolate the results", as was suggested in [10], [11].

The acquired 5D cyclic structured state–space model matrices are:

$$\dot{\mathbf{x}}(i_1,\cdots,i_5) = \mathbf{A}\mathbf{x}(i_1,\cdots,i_5)+\mathbf{b}u(i_1,\cdots,i_5) \quad (6)$$
$$y(i_1,\cdots,i_5) = \mathbf{c}'\mathbf{x}(i_1,\cdots,i_5)+du(i_1,\cdots,i_5) \quad (7)$$

where, $\dot{x}(i_1,\cdots,i_5)$ and $x(i_1,\cdots,i_5)$ are given in (3), (4), and the matrices $\mathbf{A},\mathbf{b},\mathbf{c}'$, and the scalar d are depicted in (8), (top, next page).

The dimensions of the above matrices $\mathbf{A},\mathbf{b},\mathbf{c}'$, of the above 5D state–space model, are respectively: $(5n\times 5n)$, $(5n\times 1)$, $(1\times 5n)$, and d is a scalar.

Prime examples that illustrate the theoretical concepts above presented in this work, follow below:

A. Example: First–order 5D IIR lattice discrete filter

The 5D state–space realization associated with the output $y(i_1,\cdots,i_5)$ of the first–order circuit implementation shown in Fig. 1 has the form:

$$\dot{\mathbf{x}}(i_1,\cdots,i_5) = \mathbf{A}\mathbf{x}(i_1,\cdots,i_5)+\mathbf{b}u(i_1,\cdots,i_5) \quad (9)$$
$$y(i_1,\cdots,i_5) = \mathbf{c}'\mathbf{x}(i_1,\cdots,i_5)+du(i_1,\cdots,i_5) \quad (10)$$

where,

$$\mathbf{x}(i_1,\cdots,i_5) = \begin{bmatrix} x_1^{d_1}(i_1,\cdots,i_5)\\ x_1^{d_2}(i_1,\cdots,i_5)\\ x_1^{d_3}(i_1,\cdots,i_5)\\ x_1^{d_4}(i_1,\cdots,i_5)\\ x_1^{d_5}(i_1,\cdots,i_5)\end{bmatrix},$$

$$\dot{\mathbf{x}}(i_1,\cdots,i_5) = \begin{bmatrix} x_1^{d_1}(i_1+1,i_2,\cdots,i_5)\\ x_1^{d_2}(i_1,i_2+1,\cdots,i_5)\\ x_1^{d_3}(i_1,i_2,i_3+1,i_4,i_5)\\ x_1^{d_4}(i_1,\cdots,i_3,i_4+1,i_5)\\ x_1^{d_5}(i_1,\cdots,i_4,i_5+1)\end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} -\Delta_1\Delta_2 & 1-\Delta_2^2 & 0 & 0 & 0\\ -\Delta_1\Delta_3 & -\Delta_2\Delta_3 & 1-\Delta_3^2 & 0 & 0\\ -\Delta_1\Delta_4 & -\Delta_2\Delta_4 & -\Delta_3\Delta_4 & 1-\Delta_4^2 & 0\\ -\Delta_1\Delta_5 & -\Delta_2\Delta_5 & -\Delta_3\Delta_5 & -\Delta_4\Delta_5 & 1-\Delta_5^2\\ -\Delta_1 & -\Delta_2 & -\Delta_3 & -\Delta_4 & \Delta_5\end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix}\Delta_2\\ \Delta_3\\ \Delta_4\\ \Delta_5\\ 1\end{bmatrix},$$

$$\mathbf{c}' = \begin{bmatrix} 1-\Delta_1^2 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$d = \Delta_1.$$

The dimensions of the above matrices $\mathbf{A},\mathbf{b},\mathbf{c}'$ of the above 5D state–space model, are: (5×5), (5×1), (1×5), and d is a scalar.

Applying the 5D z–transform on (9), (10),

$$H_1(z_1,z_2,z_3,z_4,z_5) = \mathbf{c}'[\mathcal{Z}-\mathbf{A}]^{-1}\mathbf{b}, \quad (11)$$

$$\mathbf{A} = \begin{bmatrix} -\Delta_1\Delta_2 & 1-\Delta_2^2 & \cdots & 0 & 0 & 0 \\ -\Delta_1\Delta_3 & -\Delta_2\Delta_3 & 1-\Delta_3^2 & 0 & 0 & 0 \\ -\Delta_1\Delta_4 & -\Delta_2\Delta_4 & -\Delta_3\Delta_4 & 1-\Delta_4^2 & 0 & 0 \\ \cdots & \cdots & \ddots & \ddots & \cdots & 0 \\ -\Delta_1\Delta_{5n-1} & \cdots & \cdots & -\Delta_{5n-2}\Delta_{5n-1} & \ddots & \\ -\Delta_1\Delta_{5n} & -\Delta_2\Delta_{5n} & \cdots & -\Delta_{5n-2}\Delta_{5n} & -\Delta_{5n-1}\Delta_{5n} & 1-\Delta_{5n}^2 \\ -\Delta_1 & -\Delta_2 & \cdots & -\Delta_{5n-2} & -\Delta_{5n-1} & -\Delta_{5n} \end{bmatrix}, \quad (8)$$

$$\mathbf{b} = \begin{bmatrix} \Delta_2 & \Delta_3 & \Delta_4 & \cdots & \Delta_{5n-1} & \Delta_{5n} & 1 \end{bmatrix}',$$

$$\mathbf{c}' = \begin{bmatrix} 1-\Delta_1^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$d = \Delta_1.$$

where, $\mathcal{Z} = diag(z_1, z_2, z_3, z_4, z_5)$. The corresponding 5D transfer–function $H_1(z_1, z_2, z_3, z_4, z_5)$ takes the form depicted in Table I.

FIRST–ORDER 5D IIR LATTICE TRANSFER FUNCTION

Variables	Numerator	Denominator
00000: constant	1	Δ_1
00001: z_5	Δ_5	$\Delta_1\Delta_5$
00010: z_4	$\Delta_4\Delta_5$	$\Delta_1\Delta_4\Delta_5$
00011: z_4z_5	Δ_4	$\Delta_1\Delta_4$
00100: z_3	$\Delta_3\Delta_4$	$\Delta_1\Delta_3\Delta_4$
00101: z_3z_5	$\Delta_3\Delta_4\Delta_5$	$\Delta_1\Delta_3\Delta_4\Delta_5$
00110: z_3z_4	$\Delta_3\Delta_5$	$\Delta_1\Delta_3\Delta_5$
00111: $z_3z_4z_5$	Δ_3	$\Delta_1\Delta_3$
01000: z_2	$\Delta_2\Delta_3$	$\Delta_1\Delta_2\Delta_3$
01001: z_2z_5	$\Delta_2\Delta_3\Delta_5$	$\Delta_1\Delta_2\Delta_3\Delta_5$
01010: z_2z_4	$\Delta_2\Delta_3\Delta_4\Delta_5$	$\Delta_1\Delta_2\Delta_3\Delta_4\Delta_5$
01011: $z_2z_4z_5$	$\Delta_2\Delta_3\Delta_4$	$\Delta_1\Delta_2\Delta_3\Delta_4$
01100: z_2z_3	$\Delta_2\Delta_4$	$\Delta_1\Delta_2\Delta_4$
01101: $z_2z_3z_5$	$\Delta_2\Delta_4\Delta_5$	$\Delta_1\Delta_2\Delta_4\Delta_5$
01110: $z_2z_3z_4$	$\Delta_2\Delta_5$	$\Delta_1\Delta_2\Delta_5$
01111: $z_2z_3z_4z_5$	Δ_2	$\Delta_1\Delta_2$
10000: z_1	$\Delta_1\Delta_2$	Δ_2
10001: z_1z_5	$\Delta_1\Delta_2\Delta_5$	$\Delta_2\Delta_5$
10010: z_1z_4	$\Delta_1\Delta_2\Delta_4\Delta_5$	$\Delta_2\Delta_4\Delta_5$
10011: $z_1z_4z_5$	$\Delta_1\Delta_2\Delta_4$	$\Delta_2\Delta_4$
10100: z_1z_3	$\Delta_1\Delta_2\Delta_3\Delta_4$	$\Delta_2\Delta_3\Delta_4$
10101: $z_1z_3z_5$	$\Delta_1\Delta_2\Delta_3\Delta_4\Delta_5$	$\Delta_2\Delta_3\Delta_4\Delta_5$
10110: $z_1z_3z_4$	$\Delta_1\Delta_2\Delta_3\Delta_5$	$\Delta_2\Delta_3\Delta_5$
10111: $z_1z_3z_4z_5$	$\Delta_1\Delta_2\Delta_3$	$\Delta_2\Delta_3$
11000: z_1z_2	$\Delta_1\Delta_3$	Δ_3
11001: $z_1z_2z_5$	$\Delta_1\Delta_3\Delta_5$	$\Delta_3\Delta_5$
11010: $z_1z_2z_4$	$\Delta_1\Delta_3\Delta_4\Delta_5$	$\Delta_3\Delta_4\Delta_5$
11011: $z_1z_2z_4z_5$	$\Delta_1\Delta_3\Delta_4$	$\Delta_3\Delta_4$
11100: $z_1z_2z_3$	$\Delta_1\Delta_4$	Δ_4
11101: $z_1z_2z_3z_5$	$\Delta_1\Delta_4\Delta_5$	$\Delta_4\Delta_5$
11110: $z_1z_2z_3z_4$	$\Delta_1\Delta_5$	Δ_5
11111: $z_1z_2z_3z_4z_5$	Δ_1	1

Table I: 5D transfer function: $H_1(z_1, z_2, z_3, z_4, z_5)$

Note that in the above 5D transfer function (Table I), the numerator and denominator coefficients are inverse as in conventional 1D all–pass lattice filters [3], [4]. Lattice–structured digital filters, with multiplier components less than unity, have discrete characteristics related to their stability as well as their quantization properties [3], [4].

B. Example: First–order 5D all–pole lattice discrete filter

The 5D state–space realization associated with the output $y(i_1, \cdots, i_5)$ of the first–order circuit implementation shown in Fig. 3 has the form:

$$\dot{\mathbf{x}}(i_1, \cdots, i_5) = \mathbf{A}\mathbf{x}(i_1, \cdots, i_5) + \mathbf{b}u(i_1, \cdots, i_5) \quad (12)$$

$$y(i_1, \cdots, i_5) = \mathbf{c}'\mathbf{x}(i_1, \cdots, i_5) + du(i_1, \cdots, i_5) \quad (13)$$

where,

$$\mathbf{x}(i_1, \cdots, i_5) = \begin{bmatrix} x_1^{d_1}(i_1, \cdots, i_5) \\ x_1^{d_2}(i_1, \cdots, i_5) \\ x_1^{d_3}(i_1, \cdots, i_5) \\ x_1^{d_4}(i_1, \cdots, i_5) \\ x_1^{d_5}(i_1, \cdots, i_5) \end{bmatrix},$$

$$\dot{\mathbf{x}}(i_1, \cdots, i_5) = \begin{bmatrix} x_1^{d_1}(i_1+1, i_2, \cdots, i_5) \\ x_1^{d_2}(i_1, i_2+1, \cdots, i_5) \\ x_1^{d_3}(i_1, i_2, i_3+1, i_4, i_5) \\ x_1^{d_4}(i_1, \cdots, i_3, i_4+1, i_5) \\ x_1^{d_5}(i_1, \cdots, i_4, i_5+1) \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} -\Delta_1\Delta_2 & 1-\Delta_2^2 & 0 & 0 & 0 \\ -\Delta_1\Delta_3 & -\Delta_2\Delta_3 & 1-\Delta_3^2 & 0 & 0 \\ -\Delta_1\Delta_4 & -\Delta_2\Delta_4 & -\Delta_3\Delta_4 & 1-\Delta_4^2 & 0 \\ -\Delta_1\Delta_5 & -\Delta_2\Delta_5 & -\Delta_3\Delta_5 & -\Delta_4\Delta_5 & 1-\Delta_5^2 \\ -\Delta_1 & -\Delta_2 & -\Delta_3 & -\Delta_4 & \Delta_5 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ 1 \end{bmatrix},$$

$$\mathbf{c}' = \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 \end{bmatrix},$$

$$d = 1.$$

The dimensions of the above matrices $\mathbf{A}, \mathbf{b}, \mathbf{c}'$ of the above first–order 5D state–space model, respectively are the following: (5×5), (5×1), (1×5), and d is a scalar.

Applying the 5D z–transform on (12), (13),

$$H_2(z_1, z_2, z_3, z_4, z_5) = \mathbf{c}'[\mathcal{Z} - \mathbf{A}]^{-1}\mathbf{b}, \quad (14)$$

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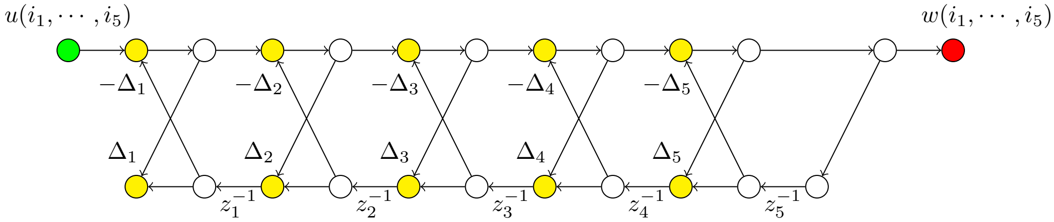

Fig. 3: 5D first–order all–pole digital lattice filter.

where, $\mathcal{Z} = diag(z_1, z_2, z_3, z_4, z_5)$. The corresponding 5D transfer–function $H_2(z_1, z_2, z_3, z_4, z_5)$ takes the form depicted in Table II.

FIRST–ORDER 5D ALL–POLE LATTICE TRANSFER FUNCTION

Variables	Numerator	Denominator
00000: constant	0	Δ_1
00001: z_5	0	$\Delta_1\Delta_5$
00010: z_4	0	$\Delta_1\Delta_4\Delta_5$
00011: z_4z_5	0	$\Delta_1\Delta_4$
00100: z_3	0	$\Delta_1\Delta_3\Delta_4$
00101: z_3z_5	0	$\Delta_1\Delta_3\Delta_4\Delta_5$
00110: z_3z_4	0	$\Delta_1\Delta_3\Delta_5$
00111: $z_3z_4z_5$	0	$\Delta_1\Delta_3$
01000: z_2	0	$\Delta_1\Delta_2\Delta_3$
01001: z_2z_5	0	$\Delta_1\Delta_2\Delta_3\Delta_5$
01010: z_2z_4	0	$\Delta_1\Delta_2\Delta_3\Delta_4\Delta_5$
01011: $z_2z_4z_5$	0	$\Delta_1\Delta_2\Delta_3\Delta_4$
01100: z_2z_3	0	$\Delta_1\Delta_2\Delta_4$
01101: $z_2z_3z_5$	0	$\Delta_1\Delta_2\Delta_4\Delta_5$
01110: $z_2z_3z_4$	0	$\Delta_1\Delta_2\Delta_5$
01111: $z_2z_3z_4z_5$	0	$\Delta_1\Delta_2$
10000: z_1	0	Δ_2
10001: z_1z_5	0	$\Delta_2\Delta_5$
10010: z_1z_4	0	$\Delta_2\Delta_4\Delta_5$
10011: $z_1z_4z_5$	0	$\Delta_2\Delta_4$
10100: z_1z_3	0	$\Delta_2\Delta_3\Delta_4$
10101: $z_1z_3z_5$	0	$\Delta_2\Delta_3\Delta_4\Delta_5$
10110: $z_1z_3z_4$	0	$\Delta_2\Delta_3\Delta_5$
10111: $z_1z_3z_4z_5$	0	$\Delta_2\Delta_3$
11000: z_1z_2	0	Δ_3
11001: $z_1z_2z_5$	0	$\Delta_3\Delta_5$
11010: $z_1z_2z_4$	0	$\Delta_3\Delta_4\Delta_5$
11011: $z_1z_2z_4z_5$	0	$\Delta_3\Delta_4$
11100: $z_1z_2z_3$	0	Δ_4
11101: $z_1z_2z_3z_5$	0	$\Delta_4\Delta_5$
11110: $z_1z_2z_3z_4$	0	Δ_5
11111: $z_1z_2z_3z_4z_5$	1	1

Table II: 5D transfer function: $H_2(z_1, z_2, z_3, z_4, z_5)$

Considering $w(i_1, \cdots, i_5)$, as the output of the filter shown in Fig. 3, its generalized transfer function for the 5D all–pole lattice filter is given in Table II.

IV. Conclusion

New circuit implementations and the corresponding state–space for 5D IIR, and all–pole digital filters were discussed. The state–space realization vectors, and the required delay components are absolutely minimum $(5n)$, the number of multipliers and adders are both $10n$. Furthermore, the resulting transfer functions are symmetric and recursive depending on their dimensionality. This recent work proves the conventional lattice filter reflective coefficient property is valid for high dimensional filters and systems. Since it has been established here in this paper that the all–pass property holds for a lattice filter up to 5D, it will be very interesting to determine the generalized structure of the transfer function for a multidimensional system. Although currently, it is now extremely difficult to do so. Thus these generalized expressions of the transfer function coefficients will be able to provide paths for more efficient circuit implementations, and state–space realizations in all dimensions now and in the future.

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